CHAPTER 4

Wave Propagation and Polarization

4.1 INTRODUCTION

In Chapter 3 we developed the vector wave equations for the electric and magnetic fields in lossless and lossy media. Solutions to the wave equations were also demonstrated in rectangular, cylindrical, and spherical coordinates using the method of *separation of variables*. In this chapter we want to consider solutions for the electric and magnetic fields of time-harmonic waves that travel in infinite lossless and lossy media. In particular, we want to develop expressions for *transverse electromagnetic* (TEM) waves (or modes) traveling along principal axes and oblique angles. The parameters of wave impedance, phase and group velocities, and power and energy densities will be discussed for each.

The concept of wave polarization will be introduced, and the necessary and sufficient conditions to achieve linear, circular, and elliptical polarizations will be discussed and illustrated. The sense of rotation, clockwise (right-hand) or counterclockwise (left-hand), will also be introduced.

4.2 TRANSVERSE ELECTROMAGNETIC MODES

A *mode* is a particular field configuration. For a given electromagnetic boundary-value problem, many field configurations that satisfy the wave equations, Maxwell's equations, and the boundary conditions usually exist. All these different field configurations (solutions) are usually referred to as *modes*.

A TEM mode is one whose field intensities, both **E** (electric) and **H** (magnetic), at every point in space are contained on a local plane, referred to as *equiphase plane*, that is independent of time. In general, the orientations of the local planes associated with the TEM wave are different at different points in space. In other words, at point (x_1, y_1, z_1) all the field components are contained on a plane. At another point (x_2, y_2, z_2) all field components are again contained on a plane; however, the two planes need not be parallel. This is illustrated in Figure 4-1*a*.

If the space orientation of the planes for a TEM mode is the same (equiphase planes are parallel), as shown in Figure 4-1*b*, then the fields form *plane waves*. In other words, the equiphase surfaces are parallel planar surfaces. If in addition to having planar equiphases the field has equiamplitude planar surfaces (the amplitude is the same over each plane), then it is called a *uniform plane wave*; that is, the field is not a function of the coordinates that form the equiphase and equiamplitude planes.

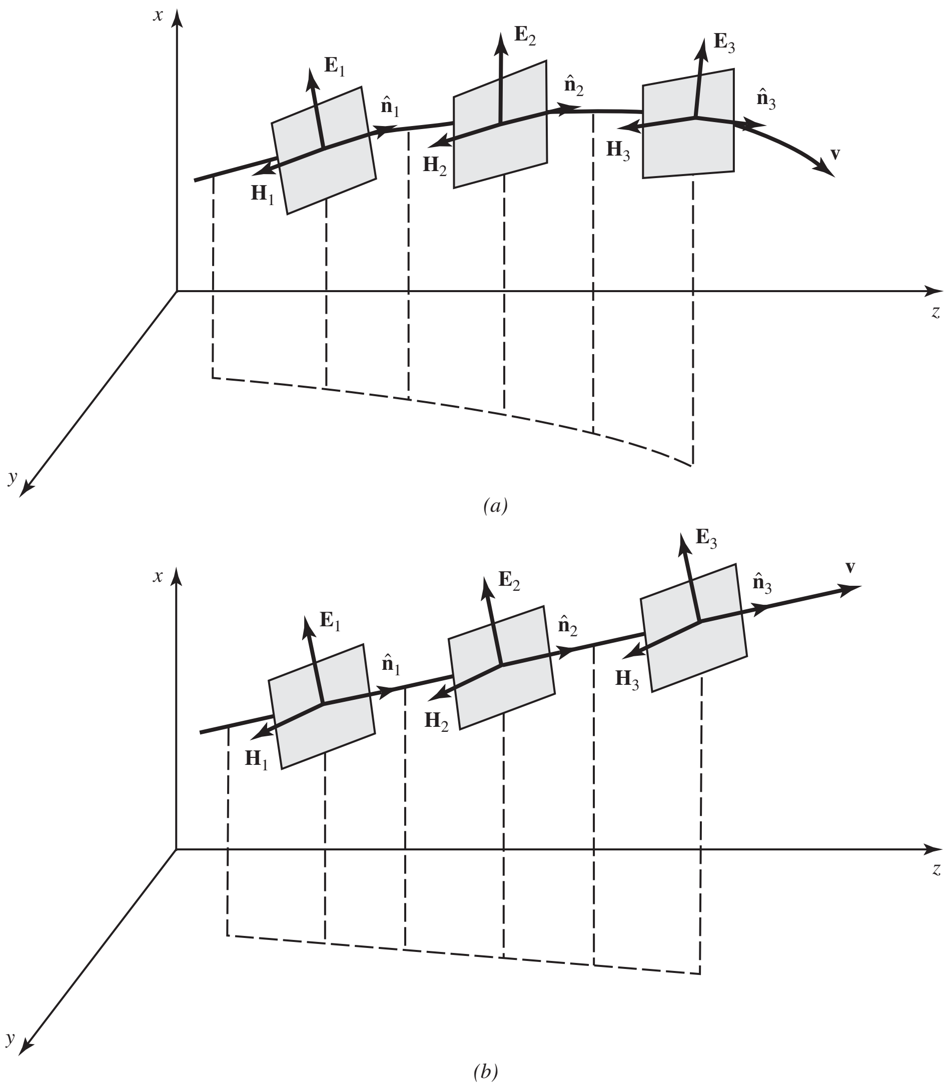
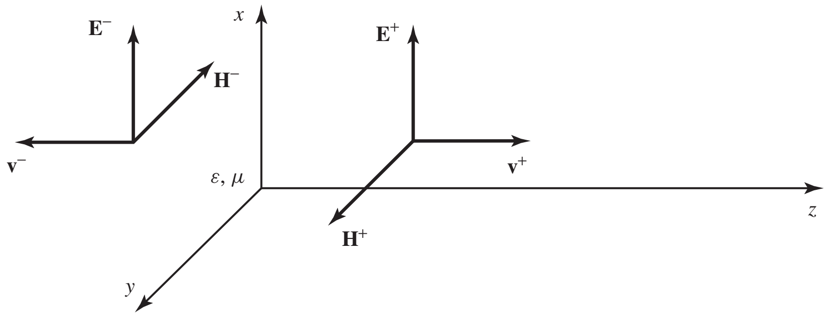


Figure 4-1 Phase fronts of waves. (*a*) TEM. (*b*) Plane.

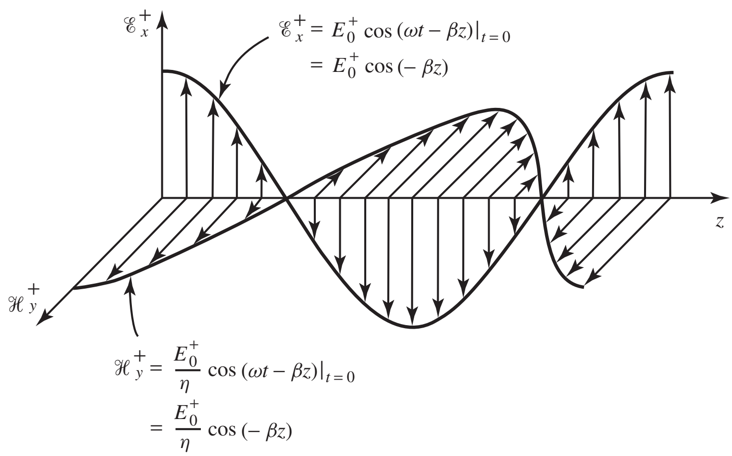
4.2.1 Uniform Plane Waves in an Unbounded Lossless Medium — Principal Axis

In this section we will write expressions for the electric and magnetic fields of a uniform plane wave traveling in an unbounded medium. In addition the wave impedance, phase and energy (group) velocities, and power and energy densities of the wave will be discussed.

A. Electric and Magnetic Fields Let us assume that a time-harmonic uniform plane wave is traveling in an unbounded lossless medium (ε, μ) in the z direction (either positive or negative), as shown in Figure 4-2a. In addition, for simplicity, let us assume the electric field of the wave has only an x component. We want to write expressions for the electric and magnetic fields associated with this wave.



(a)



(b)

Figure 4-2 Uniform plane wave fields. (*a*) Complex. (*b*) Instantaneous.

For the electric and magnetic field components to be valid solutions of a time-harmonic electromagnetic wave, they must satisfy Maxwell's equations as given in Table 1-4 or the corresponding wave equations as given, respectively, by (3-18a) and (3-18b). Here the approach will be to initiate the solution by solving the wave equation for either the electric or magnetic field and then finding the other field using Maxwell's equations. An alternate procedure, which has been assigned as an end-of-chapter problem, would be to follow the entire solution using only Maxwell's equations.

Since the electric field has only an x component, it must satisfy the scalar wave equation of (3-20a) or (3-22), whose general solution is given by (3-23). Because the wave is a uniform plane wave that travels in the z direction, its solution is not a function of x and y. Therefore (3-23) reduces to

$$E_x(z) = h(z) \tag{4-1}$$

The solutions of $h(z)$ are given by (3-30a) or (3-30b). Since the wave in question is a traveling wave, instead of a standing wave, its most appropriate solution is that given by (3-30a). The first term in (3-30a) represents a wave that travels in the $+z$ direction and the second term represents a wave that travels in the $-z$ direction. Therefore the solution of (4-1), using (3-30a), can be written as

$$E_x(z) = A_3 e^{-j\beta z} + B_3 e^{+j\beta z} = E_x^+ + E_x^- \tag{4-2}$$

or

$$E_x(z) = E_0^+ e^{-j\beta z} + E_0^- e^{+j\beta z} = E_x^+ + E_x^- \tag{4-2a}$$

$$E_x^+(z) = E_0^+ e^{-j\beta z} \tag{4-2b}$$

$$E_x^-(z) = E_0^- e^{+j\beta z} \tag{4-2c}$$

since $\beta_z = \beta$ because $\beta_x = \beta_y = 0$. E_0^+ and E_0^- represent, respectively, the amplitudes of the positive and negative (in the z direction) traveling waves.

The corresponding magnetic field must also be a solution of its wave equation 3-18b, and its form will be similar to (4-2). However, since we do not know which components of magnetic field coexist with the x component of the electric field, they are most appropriately determined by using one of Maxwell's equations as given in Table 1-4. Since the electric field is known, as given by (4-2), the magnetic field can best be found using

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{4-3}$$

or

$$\mathbf{H} = -\frac{1}{j\omega\mu}\nabla \times \mathbf{E} = -\frac{1}{j\omega\mu}\begin{bmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & 0 & 0 \end{bmatrix} \tag{4-3a}$$

which, using (4-2a), reduces to

$$\mathbf{H} = -\hat{\mathbf{a}}_y \frac{1}{j\omega\mu}\left\{\frac{\partial E_x}{\partial z}\right\} = \hat{\mathbf{a}}_y \frac{\beta}{\omega\mu}\left\{E_0^+ e^{-j\beta z} - E_0^- e^{+j\beta z}\right\}$$

$$\mathbf{H} = \hat{\mathbf{a}}_y \frac{1}{\sqrt{\mu/\varepsilon}}\left\{E_0^+ e^{-j\beta z} - E_0^- e^{+j\beta z}\right\} = \hat{\mathbf{a}}_y \frac{1}{\sqrt{\mu/\varepsilon}}\left\{E_x^+ - E_x^-\right\} = \hat{\mathbf{a}}_y \left\{H_y^+ + H_y^-\right\} \tag{4-3b}$$

where

$$H_y^+ = \frac{1}{\sqrt{\mu/\varepsilon}}E_x^+ \tag{4-3c}$$

$$H_y^- = -\frac{1}{\sqrt{\mu/\varepsilon}}E_x^- \tag{4-3d}$$

Plots of the instantaneous *positive* traveling electric and magnetic fields at $t = 0$ as a function of z are shown in Figure 4-2b. Similar plots can be drawn for the negative traveling fields.

B. Wave Impedance Since each term for the magnetic field (A/m) in (4-3c) and (4-3d) is individually identical to the corresponding term for the electric field (V/m) in (4-2a), the factor $\sqrt{\mu/\varepsilon}$ in the denominator in (4-3c) and (4-3d) must have units of ohms (V/A). Therefore the factor $\sqrt{\mu/\varepsilon}$ is known as the *wave impedance*, Z_w, denoted by the ratio of the electric to magnetic field, and it is usually represented by η

$$Z_w = \frac{E_x^+}{H_y^+} = -\frac{E_x^-}{H_y^-} = \eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{4-4}$$

The wave impedance of (4-4) is identical to a quantity that is referred to as the *intrinsic impedance* $\eta = \sqrt{\mu/\varepsilon}$ of the medium. In general, this is true not only for uniform plane waves but also for plane and TEM waves; however, it is not true for TE or TM modes.

In (4-3d) it is also observed that a negative sign is found in front of the magnetic field component that travels in the $-z$ direction; a positive sign is noted in front of the positive traveling wave. The general procedure that can be followed to find the magnetic field components, given the electric field components, or to find the electric field components, given the magnetic field components, is the following:

1. Place the fingers of your right hand in the direction of the electric field component.
2. Direct your thumb toward the direction of wave travel (power flow).
3. Rotate your fingers 90° in a direction so that a right-hand screw is formed.
4. The new direction of your fingers is the direction of the magnetic field component.
5. Divide the electric field component by the wave impedance to obtain the corresponding magnetic field component.

The foregoing procedure must be followed for each term of each component of an electric or magnetic field. The results are identical to those that would be obtained by using Maxwell's equations. If the wave impedance is known in advance, as it is for TEM waves, this procedure is simpler and much more rapid than using Maxwell's equations. By following this procedure, the answers (including the signs) in (4-3c) and (4-3d) given (4-2b) and (4-2c) are obvious.

To illustrate the procedure, let us consider another example.

Example 4-1

The electric field of a uniform plane wave traveling in free space is given by

$$\mathbf{E} = \hat{\mathbf{a}}_y\left(E_0^+ e^{-j\beta z} + E_0^- e^{+j\beta z}\right) = \hat{\mathbf{a}}_y\left(E_y^+ + E_y^-\right)$$

where E_0^+ and E_0^- are constants. Find the corresponding magnetic field using the outlined procedure.

Solution: For the electric field component that is traveling in the $+z$ direction, the corresponding magnetic field component is given by

$$\mathbf{H}^+ = -\hat{\mathbf{a}}_x \frac{E_0^+}{\eta_0} e^{-j\beta z} \simeq -\hat{\mathbf{a}}_x \frac{E_0^+}{377} e^{-j\beta z}$$

where

$$\eta_0 = Z_w = \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 377\,\text{ohms}$$

Similarly, for the wave that is traveling in the $-z$ direction we can write that

$$\mathbf{H}^- = \hat{\mathbf{a}}_x \frac{E_0^-}{\eta_0} e^{+j\beta z} \simeq \hat{\mathbf{a}}_x \frac{E_0^-}{377} e^{+j\beta z}$$

Therefore the total magnetic field is equal to

$$\mathbf{H} = \mathbf{H}^+ + \mathbf{H}^- = \hat{\mathbf{a}}_x \frac{1}{\eta_0}\left(-E_0^+ e^{-j\beta z} + E_0^- e^{+j\beta z}\right)$$

The same answer would be obtained if Maxwell's equations were used, and it is assigned as an end-of-chapter problem.

The term in the expression for the electric field in (4-2a) that identifies the direction of wave travel can also be written in vector notation. This is usually more convenient to use when dealing with waves traveling at oblique angles. Equation 4-2a can therefore take the more general form of

$$E_x(z) = E_0^+ e^{-j\boldsymbol{\beta}^+ \cdot \mathbf{r}} + E_0^- e^{-j\boldsymbol{\beta}^- \cdot \mathbf{r}} \tag{4-5}$$

where

$$\boldsymbol{\beta}^+ = \hat{\boldsymbol{\beta}}^+ \beta = \hat{\mathbf{a}}_x \beta_x^+ + \hat{\mathbf{a}}_y \beta_y^+ + \hat{\mathbf{a}}_z \beta_z^+ \Big|_{\substack{\beta_x^+ = \beta_y^+ = 0 \\ \beta_z^+ = \beta}} = \hat{\mathbf{a}}_z \beta \tag{4-5a}$$

$$\boldsymbol{\beta}^- = \hat{\boldsymbol{\beta}}^- \beta = \hat{\mathbf{a}}_x \beta_x^- + \hat{\mathbf{a}}_y \beta_y^- - \hat{\mathbf{a}}_z \beta_z^- \Big|_{\substack{\beta_x^- = \beta_y^- = 0 \\ \beta_z^- = \beta}} = -\hat{\mathbf{a}}_z \beta \tag{4-5b}$$

$$\mathbf{r} = \text{position vector} = \hat{\mathbf{a}}_x x + \hat{\mathbf{a}}_y y + \hat{\mathbf{a}}_z z \tag{4-5c}$$

In (4-5a) through (4-5c), $\beta_x, \beta_y, \beta_z$ represent, respectively, the phase constants of the wave in the x, y, z directions, $\mathbf{r}$ represents the position vector in rectangular coordinates, and $\hat{\boldsymbol{\beta}}^+$ and $\hat{\boldsymbol{\beta}}^-$ represent unit vectors in the directions of $\boldsymbol{\beta}^+$ and $\boldsymbol{\beta}^-$. The notation used in (4-5) through (4-5c) to represent the wave travel will be most convenient to express wave travel at oblique angles, as will be the case in Section 4.2.2.

C. Phase and Energy (Group) Velocities, Power, and Energy Densities The expressions for the electric and magnetic fields, as given by (4-2a) and (4-3b), represent the spatial variations of the field intensities. The corresponding instantaneous forms of each can be written, using (1-61a) and (1-61b) and assuming E_0^+ and E_0^- are real constants, as

$$\begin{aligned}\mathscr{E}_x(z;t) &= \mathscr{E}_x^+(z;t) + \mathscr{E}_x^-(z;t) = \operatorname{Re}\left[E_0^+ e^{-j\beta z} e^{j\omega t}\right] + \operatorname{Re}\left[E_0^- e^{+j\beta z} e^{j\omega t}\right] \\ &= E_0^+ \cos(\omega t - \beta z) + E_0^- \cos(\omega t + \beta z) \end{aligned}\tag{4-6a}$$

$$\begin{aligned}\mathscr{H}_y(z;t) &= \mathscr{H}_y^+(z;t) + \mathscr{H}_y^-(z;t) \\ &= \frac{1}{\sqrt{\mu/\varepsilon}}\left[E_0^+ \cos(\omega t - \beta z) - E_0^- \cos(\omega t + \beta z)\right] \end{aligned}\tag{4-6b}$$

In each of the fields, as given by (4-6a) and (4-6b), the first term represents, according to (3-34) through (3-35) and Figure 3-3, a wave that travels in the $+z$ direction; the second term represents a wave that travels in the $-z$ direction. To maintain a constant phase in the first term of (4-6a), the velocity must be equal, according to (3-35), to

$$v_p^+ = +\frac{dz}{dt} = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{4-7}$$

The corresponding velocity of the second term in (4-6a) is identical in magnitude to (4-7) but with a negative sign to reflect the direction of wave travel. The velocity of (4-7) is referred to as the *phase velocity*, and it represents the velocity that must be maintained in order to keep in step with a constant phase front of the wave. As will be shown for oblique traveling waves, the phase velocity of such waves can exceed the velocity of light. This is only a hypothetical speed, as will be explained in Section 4.2.2C. Aside of nonuniform plane waves, also referred to as slow surface waves (see Section 5.3.4A), in general the phase velocity can be equal to or even greater than the speed of light. Variations of the instantaneous positive traveling electric $\mathscr{E}_x^+(z;t)$ and magnetic $\mathscr{H}_y^+(z;t)$ fields as a function of z for $t=0$ are shown in Figure 4-2b. As time increases, both curves will shift in the positive z direction. A similar set of curves can be drawn for the negative traveling electric $\mathscr{E}_x^-(z;t)$ and magnetic $\mathscr{H}_y^-(z;t)$ fields.

The electric and magnetic energies (W-s/m^3) and power densities (W/m^2) associated with the positive traveling waves of (4-6a) and (4-6b) can be written, according to (l-58f) and (l-58e), as

$$w_e^+ = \frac{1}{2}\varepsilon\mathscr{E}_x^{+2} = \frac{1}{2}\varepsilon E_0^{+2}\cos^2(\omega t - \beta z) \tag{4-8a}$$

$$w_m^+ = \frac{1}{2}\mu\mathscr{H}_y^{+2} = \frac{1}{2}\mu\left[(\varepsilon/\mu)\, E_0^{+2}\cos^2(\omega t - \beta z)\right] = \frac{1}{2}\varepsilon E_0^{+2}\cos^2(\omega t - \beta z) \tag{4-8b}$$

$$\begin{aligned}\boldsymbol{\mathscr{S}}^+ &= \boldsymbol{\mathscr{E}}^+ \times \boldsymbol{\mathscr{H}}^+ = \hat{\mathbf{a}}_x E_0^+\cos(\omega t - \beta z) \times \left[\hat{\mathbf{a}}_y \left(1/\sqrt{\mu/\varepsilon}\right) E_0^+\cos(\omega t - \beta z)\right] \\ &= \hat{\mathbf{a}}_z \mathscr{S}^+ = \hat{\mathbf{a}}_z \left(1/\sqrt{\mu/\varepsilon}\right) E_0^{+2}\cos^2(\omega t - \beta z)\end{aligned} \tag{4-8c}$$

The ratio formed by dividing the power density $\mathscr{S}$ (W/m^2) by the total energy density $w = w_e + w_m$ (J/m^3 = W-s/m^3) is referred to as the *energy* (*group*) *velocity* v_e, and it is given by

$$v_e^+ = \frac{\mathscr{S}^+}{w^+} = \frac{\mathscr{S}^+}{w_e^+ + w_m^+} = \frac{\left(1/\sqrt{\mu/\varepsilon}\right) E_0^{+2}\cos^2(\omega t - \beta z)}{\varepsilon E_0^{+2}\cos^2(\omega t - \beta z)} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{4-9}$$

The energy velocity represents the velocity with which the wave energy is transported. It is apparent that (4-9) is identical to (4-7). In general that is not the case. In fact, the energy velocity v_e^+ can be equal to, but not exceed, the speed of light, and the product of the phase velocity v_p and energy velocity v_e must always be equal to

$$\boxed{v_p^+ v_e^+ = (v^+)^2 = \frac{1}{\mu\varepsilon}} \tag{4-10}$$

where $v^+ = 1/\sqrt{\mu\varepsilon}$ is the speed of light. The same holds for the negative traveling waves.

The time-average power density (Poynting vector) associated with the positive traveling wave can be written, using (l-70) and the first terms of (4-2a) and (4-3b), as

$$\boldsymbol{\mathscr{S}}_{\text{av}}^+ = \frac{1}{2}\,\text{Re}\left(\mathbf{E}^+ \times \mathbf{H}^{+*}\right) = \hat{\mathbf{a}}_z \frac{1}{2\sqrt{\mu/\varepsilon}}|E_x^+|^2 = \hat{\mathbf{a}}_z \frac{|E_0^+|^2}{2\sqrt{\mu/\varepsilon}} = \hat{\mathbf{a}}_z \frac{|E_0^+|^2}{2\eta} \tag{4-11}$$

A similar expression is derived for the negative traveling wave.

D. Standing Waves Each of the terms in (4-2a) and (4-3b) represents individually *traveling* waves, the first traveling in the positive z direction and the second in the negative z direction. The two together form a so-called *standing wave*, which is comprised of two oppositely traveling waves.

To examine the characteristics of a standing wave, let us rewrite (4-2a) as

$$\begin{aligned}E_x(z) &= E_0^+ e^{-j\beta z} + E_0^- e^{+j\beta z} \\ &= E_0^+\left[\cos(\beta z) - j\sin(\beta z)\right] + E_0^-\left[\cos(\beta z) + j\sin(\beta z)\right] \\ &= \left(E_0^+ + E_0^-\right)\cos(\beta z) - j\left(E_0^+ - E_0^-\right)\sin(\beta z)\end{aligned}$$

$$E_x(z) = \sqrt{\left(E_0^+ + E_0^-\right)^2 \cos^2(\beta z) + \left(E_0^+ - E_0^-\right)^2 \sin^2(\beta z)}$$
$$\times \exp\left\{-j \tan^{-1}\left[\frac{\left(E_0^+ - E_0^-\right)\sin(\beta z)}{\left(E_0^+ + E_0^-\right)\cos(\beta z)}\right]\right\}$$
$$E_x(z) = \sqrt{\left(E_0^+\right)^2 + \left(E_0^-\right)^2 + 2E_0^+E_0^- \cos(2\beta z)}$$
$$\times \exp\left\{-j \tan^{-1}\left[\frac{\left(E_0^+ - E_0^-\right)}{\left(E_0^+ + E_0^-\right)}\tan(\beta z)\right]\right\} \tag{4-12}$$

The amplitude of the waveform given by (4-12) is equal to

$$|E_x(z)| = \sqrt{\left(E_0^+\right)^2 + \left(E_0^-\right)^2 + 2E_0^+E_0^- \cos(2\beta z)} \tag{4-12a}$$

By examining (4-12a), it is evident that its maximum and minimum values are given, respectively, by

$$|E_x(z)|_{\max} = |E_0^+| + |E_0^-| \text{ when } \beta z = m\pi, m = 0, 1, 2, \ldots \tag{4-13a}$$

and for $|E_0^+| > |E_0^-|$,

$$|E_x(z)|_{\min} = |E_0^+| - |E_0^-| \text{ when } \beta z = \frac{(2m+1)\pi}{2}, m = 0, 1, 2, \ldots \tag{4-13b}$$

Neighboring maximum and minimum values are separated by a distance of $\lambda/4$ or successive maxima or minima are separated by $\lambda/2$.

The instantaneous field of (4-12) can also be written as

$$\mathscr{E}_x(z;t) = \text{Re}\left[E_x(z)\, e^{j\omega t}\right]$$
$$= \sqrt{\left(E_0^+\right)^2 + \left(E_0^-\right)^2 + 2E_0^+E_0^- \cos(2\beta z)}$$
$$\times \cos\left[\omega t - \tan^{-1}\left\{\frac{E_0^+ - E_0^-}{E_0^+ + E_0^-}\tan(\beta z)\right\}\right] \tag{4-14}$$

It is apparent that (4-12a) represents the envelope of the maximum values the instantaneous field of (4-14) will achieve as a function of time at a given position. Since this envelope of maximum values does not move (change) in position as a function of time, it is referred to as the *standing wave pattern* and the associated wave of (4-12) or (4-14) is referred to as the *standing wave*.

The ratio of the maximum/minimum values of the standing wave pattern of (4-12a), as given by (4-13a) and (4-13b), is referred to as the standing wave ratio (SWR), and it is given by

$$\text{SWR} = \frac{|E_x(z)|_{\max}}{|E_x(z)|_{\min}} = \frac{|E_0^+| + |E_0^-|}{|E_0^+| - |E_0^-|} = \frac{1 + \frac{|E_0^-|}{|E_0^+|}}{1 - \frac{|E_0^-|}{|E_0^+|}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \tag{4-15}$$

where Γ is the reflection coefficient. Since in transmission lines we usually deal with voltages and currents (instead of electric and magnetic fields), the SWR is usually referred to as the VSWR (voltage standing wave ratio). Plots of the standing wave pattern in terms of E_0^+ as a function of $z(-\lambda \le z \le \lambda)$ for $|\Gamma| = 0$, 0.2, 0.4, 0.6, 0.8, and 1 are shown in Figure 4-3.

The SWR is a quantity that can be measured with instrumentation [1, 2]. SWR has values in the range of $1 \le \text{SWR} \le \infty$. The value of the SWR indicates the amount of interference between the two opposite traveling waves; the smaller the SWR value, the lesser the interference.

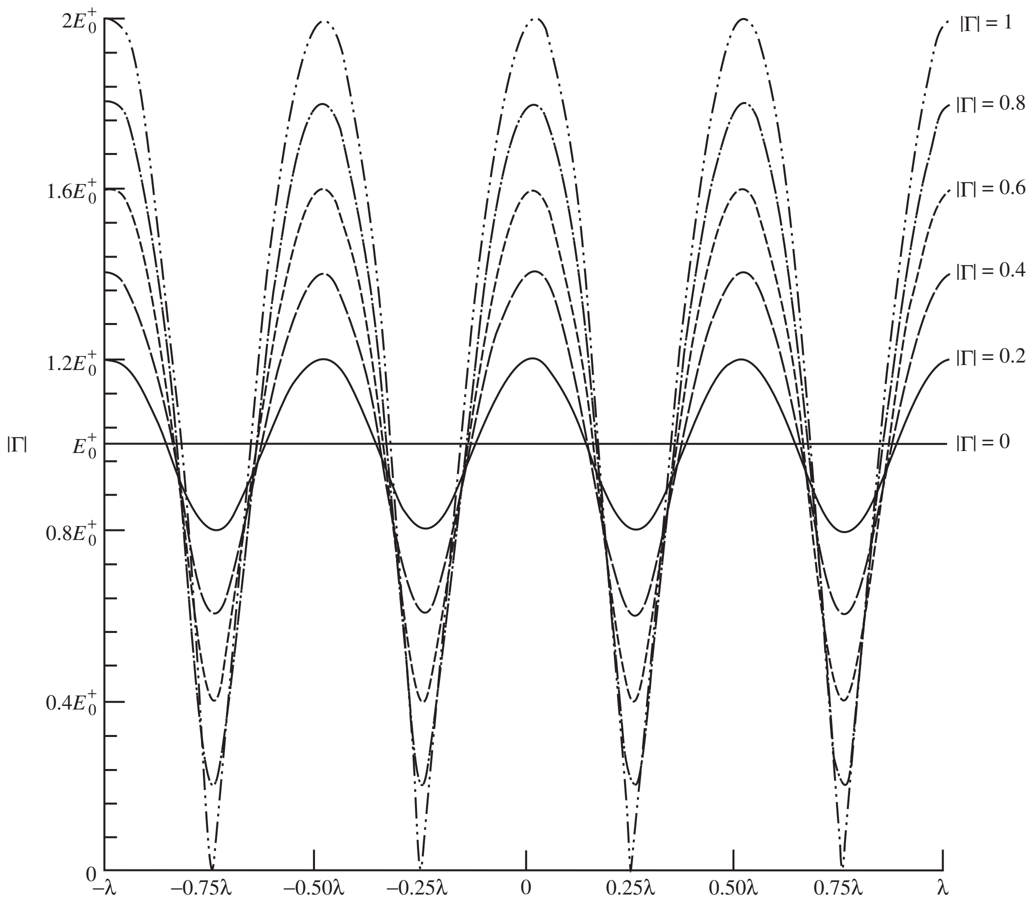


Figure 4-3 Standing wave pattern as a function of distance for a uniform plane wave with different reflection coefficients.

The minimum SWR value of unity occurs when $|\Gamma| = E_0^-/E_0^+ = 0$, and it indicates that no interference is formed. Thus the standing wave reduces to a pure traveling wave. The maximum SWR of infinity occurs when $|\Gamma| = E_0^-/E_0^+ = 1$, and it indicates that the negative traveling wave is of the same intensity as the positive traveling wave. This provides the maximum interference, and the wave forms a pure standing wave pattern given by

$$|E_x(z)|_{E_0^+ = E_0^-} = 2E_0^+ |\cos(\beta z)| = 2E_0^- |\cos(\beta z)| \tag{4-16}$$

The pattern of this is a rectified cosine function, and it is represented in Figure 4-3 by the $|\Gamma| = 1$ curve. The pattern exhibits pure nulls and peak values of twice the amplitude of the incident wave.

4.2.2 Uniform Plane Waves in an Unbounded Lossless Medium — Oblique Angle

In this section, expressions for the electric and magnetic fields, wave impedance, phase and group velocities, and power and energy densities will be written for uniform plane waves traveling at oblique angles in an unbounded medium. All of these will be done for waves that are uniform plane waves to the direction of travel.

A. Electric and Magnetic Fields Let us assume that a uniform plane wave is traveling in an unbounded medium in a direction shown in Figure 4-4*a*. The amplitudes of the positive and negative traveling electric fields are E_0^+ and E_0^-, respectively, and the assumed directions of each are also illustrated in Figure 4-4*a*. It is desirable to write expressions for the positive and negative traveling electric and magnetic field components.

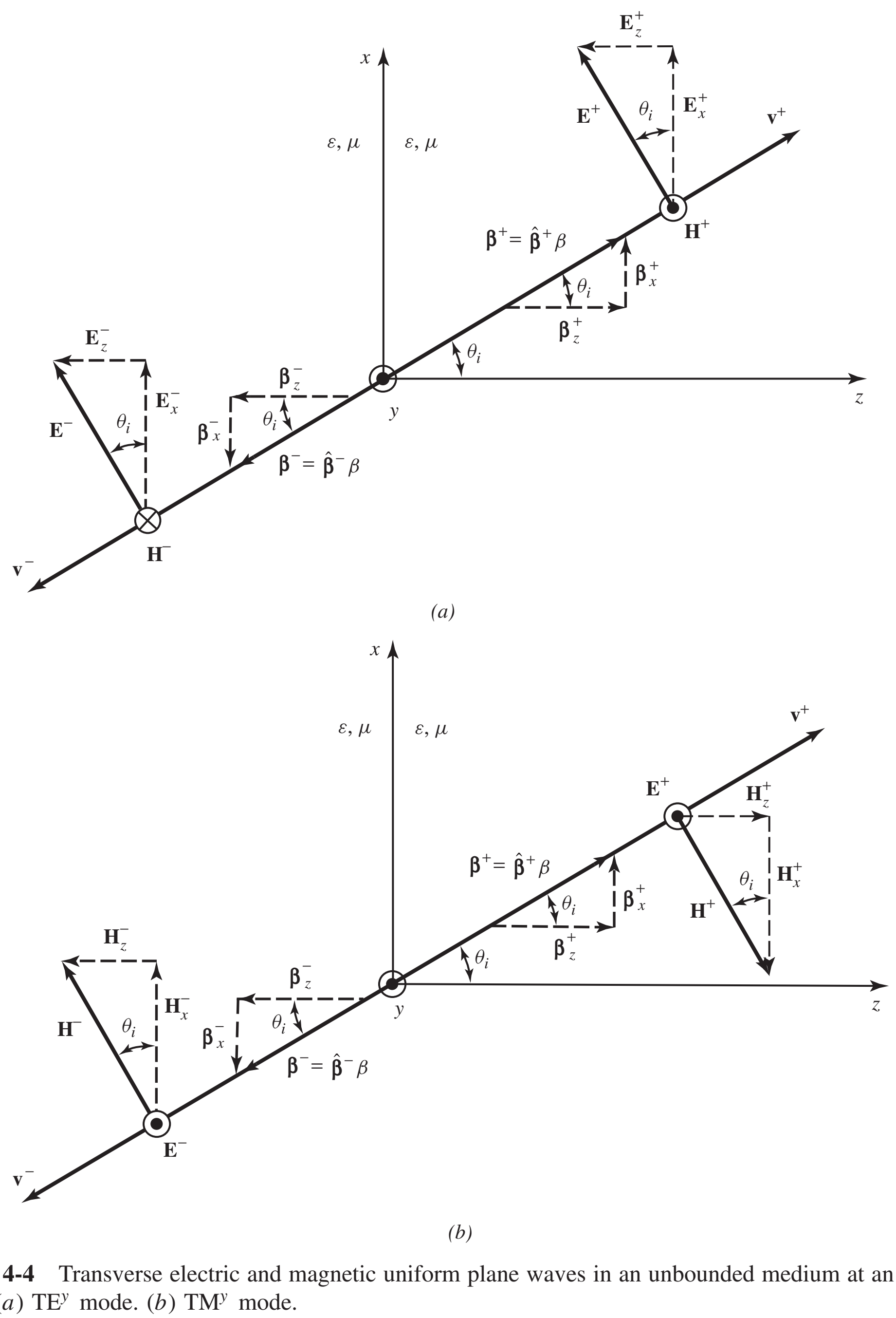


Figure 4-4 Transverse electric and magnetic uniform plane waves in an unbounded medium at an oblique angle. (*a*) TE^y mode. (*b*) TM^y mode.

Since the electric field of the wave of Figure 4-4*a* does not have a *y* component, the field configuration is referred to as *transverse electric to y* (TE^y). More detailed discussion on the construction of *transverse electric* (TE) and *transverse magnetic* (TM) field configurations, as well as *transverse electromagnetic* (TEM), can be found in Chapter 6.

Because for the TE^y wave of Figure 4-4*a* the electric field is pointing along a direction that does not coincide with any of the principal axes, it can be decomposed into components coincident with the principal axes. According to the geometry of Figure 4-4*a*, it is evident that the electric field can be written as

$$\mathbf{E} = \mathbf{E}^+ + \mathbf{E}^- = E_0^+ (\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i) e^{-j\boldsymbol{\beta}^+ \cdot \mathbf{r}} + E_0^- (\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i) e^{-j\boldsymbol{\beta}^- \cdot \mathbf{r}} \quad (4\text{-}17)$$

where $\mathbf{r}$ is the position vector of (4-5c), and it is displayed graphically in Figure 4-5. Since the phase constants $\boldsymbol{\beta}^+$ and $\boldsymbol{\beta}^-$ can be written, respectively, as

$$\boldsymbol{\beta}^+ = \hat{\boldsymbol{\beta}}^+ \beta = \hat{\mathbf{a}}_x \beta_x^+ + \hat{\mathbf{a}}_z \beta_z^+ = \beta (\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i) \quad (4\text{-}17a)$$

$$\boldsymbol{\beta}^- = \hat{\boldsymbol{\beta}}^- \beta = \hat{\mathbf{a}}_x \beta_x^- + \hat{\mathbf{a}}_z \beta_z^- = -\beta (\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i) \quad (4\text{-}17b)$$

(4-17) can be expressed as

$$\mathbf{E} = E_0^+ (\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i) e^{-j\beta(x \sin\theta_i + z \cos\theta_i)} + E_0^- (\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i) e^{+j\beta(x \sin\theta_i + z \cos\theta_i)} \quad (4\text{-}18a)$$

Since the wave is a uniform plane wave, the amplitude of its magnetic field is related to the amplitude of its electric field by the wave impedance (in this case also by the intrinsic

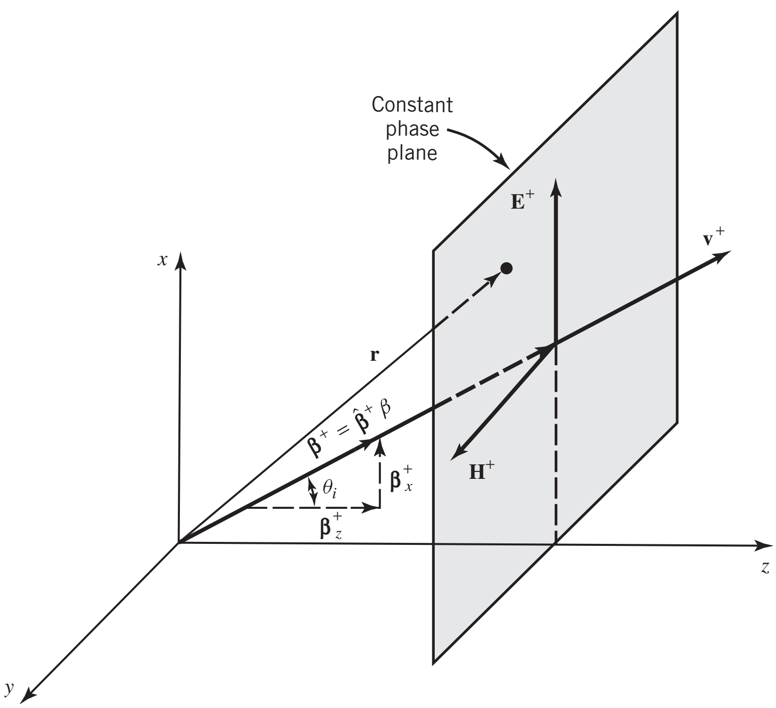


Figure 4-5 Phase front of a TEM wave traveling in a general direction.

impedance) as given by (4-4). Since the magnetic field is traveling in the same direction as the electric field, the exponentials used to indicate its directions of travel are the same as those of the electric field as given in (4-18a). The directions of the magnetic field can be found using the right-hand procedure outlined in Section 4.2.1 and illustrated graphically in Figure 4-2*b* for the positive traveling wave. Using all of the preceding information, it is evident that the magnetic field corresponding to the electric field of (4-18a) can be written as

$$\mathbf{H} = \mathbf{H}^+ + \mathbf{H}^- = \hat{\mathbf{a}}_y \left[\frac{E_0^+}{\eta} e^{-j\beta(x \sin\theta_i + z\cos\theta_i)} - \frac{E_0^-}{\eta} e^{+j\beta(x\sin\theta_i + z\cos\theta_i)} \right] \tag{4-18b}$$

In vector form, (4-18b) can also be written as

$$\mathbf{H} = \frac{1}{\eta}\left[\hat{\boldsymbol{\beta}}^+ \times \mathbf{E}^+ + \hat{\boldsymbol{\beta}}^- \times \mathbf{E}^-\right] \tag{4-18c}$$

The same form can be used to relate the **E** and **H** for any TEM wave traveling in any direction. It is apparent that when $\theta_i = 0$, (4-18a) and (4-18b) reduce to (4-2a) and (4-3b), respectively. The same answer for the magnetic field of (4-18b) can be obtained by applying Maxwell's equation 4-3 to the electric field of (4-18a). This is left for the reader as an end-of-the-chapter exercise.

The planes of constant phase at any time t are obtained by setting the phases of (4-18a) or (4-18b) equal to a constant, that is

$$\boldsymbol{\beta}^+ \cdot \mathbf{r} = \beta_x^+ x + \beta_y^+ y + \beta_z^+ z|_{y=0} = \beta\,(x\sin\theta_i + z\cos\theta_i) = C^+ \tag{4-19a}$$

$$\boldsymbol{\beta}^- \cdot \mathbf{r} = \beta_x^- x + \beta_y^- y + \beta_z^- z|_{y=0} = -\beta\,(x\sin\theta_i + z\cos\theta_i) = C^- \tag{4-19b}$$

Each of (4-19a) and (4-19b) are equations of a plane in either the spherical or rectangular coordinates with unit vectors $\hat{\boldsymbol{\beta}}^+$ and $\hat{\boldsymbol{\beta}}^-$ normal to each of the respective surfaces. The respective phase velocities in any direction (r, x, or z) are obtained by letting

$$\boldsymbol{\beta}^+ \cdot \mathbf{r} - \omega t = \beta\,(x\sin\theta_i + z\cos\theta_i) - \omega t = C_0^+ \tag{4-19c}$$

$$\boldsymbol{\beta}^- \cdot \mathbf{r} - \omega t = -\beta\,(x\sin\theta_i + z\cos\theta_i) - \omega t = C_0^- \tag{4-19d}$$

and taking a derivative with respect to time.

Example 4-2

Another exercise of interest is that in which the electric field is directed along the $+y$ direction and the wave is traveling along an oblique angle θ_i, as shown in Figure 4-4*b*. This is referred to as a TM^y wave. The objective here is again to write expressions for the positive and negative electric and magnetic field components, assuming the amplitudes of the positive and negative electric field components are E_0^+ and E_0^-, respectively.

Solution: Since this wave only has a y electric field component, and it is traveling in the same direction as that of Figure 4-4*a*, we can write the electric field as

$$\mathbf{E} = \mathbf{E}^+ + \mathbf{E}^- = \hat{\mathbf{a}}_y \left[E_0^+ e^{-j\beta(x\sin\theta_i + z\cos\theta_i)} + E_0^- e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}\right]$$

Using the right-hand procedure outlined in Section 4.2.1, the corresponding magnetic field components are pointed along directions indicated in Figure 4-4*b*. Since the magnetic field is not directed along any of the principal axes, it can be decomposed into components that coincide with the principal axes, as shown in Figure 4-4*b*. Doing this and relating the amplitude of the electric and magnetic fields by the intrinsic impedance, we can write the magnetic field as

$$\mathbf{H} = \mathbf{H}^+ + \mathbf{H}^- = \frac{E_0^+}{\eta}(-\hat{\mathbf{a}}_x \cos\theta_i + \hat{\mathbf{a}}_z \sin\theta_i)\, e^{-j\beta(x\sin\theta_i + z\cos\theta_i)}$$
$$+ \frac{E_0^-}{\eta}(\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\, e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}$$

The same answers could have been obtained if Maxwell's equation 4-3 were used. Since the magnetic field does not have any y components, this field configuration is referred to as *transverse magnetic to y* (TM^y), which will be discussed in more detail in Chapter 6.

B. Wave Impedance Since the TE^y and TM^y fields of Section 4.2.2A were TEM to the direction of travel, the wave impedance of each in the direction $\boldsymbol{\beta}$ of wave travel is the same as the intrinsic impedance of the medium. However, there are other directional impedances toward the x and z directions. These impedances are obtained by dividing the electric field component by the corresponding orthogonal magnetic field component. These two components are chosen so that the cross product of the electric field and the magnetic field, which corresponds to the direction of power flow, is in the direction of the wave travel.

Following the aforementioned procedure, the directional impedances for the TE^y fields of (4-18a) and (4-18b) can be written as

$$\underline{\mathrm{TE}^y}$$

$$Z_x^+ = -\frac{E_z^+}{H_y^+} = \eta\sin\theta_i = Z_x^- = \frac{E_z^-}{H_y^-} \tag{4-20a}$$

$$Z_z^+ = \frac{E_x^+}{H_y^+} = \eta\cos\theta_i = Z_z^- = -\frac{E_x^-}{H_y^-} \tag{4-20b}$$

In the same manner, the directional impedances of the TM^y fields of Example 4-2 can be written as

$$\underline{\mathrm{TM}^y}$$

$$Z_x^+ = \frac{E_y^+}{H_z^+} = \frac{\eta}{\sin\theta_i} = Z_x^- = -\frac{E_y^-}{H_z^-} \tag{4-21a}$$

$$Z_z^+ = -\frac{E_y^+}{H_x^+} = \frac{\eta}{\cos\theta_i} = Z_z^- = \frac{E_y^-}{H_x^-} \tag{4-21b}$$

It is apparent from the preceding results that the directional impedances of the TE^y oblique incidence traveling waves are equal to or smaller than the intrinsic impedance and those of the TM^y are equal to or larger than the intrinsic impedance. In addition, the positive and negative directional impedances of the same orientation are the same. This is the main principle of the *transverse resonance method* (see Section 8.6), which is used to analyze microwave circuits and antenna systems [3, 4].

C. Phase and Energy (Group) Velocities The wave velocity v_r of the fields given by (4-18a) and (4-18b) in the direction $\boldsymbol{\beta}$ of travel is equal to the speed of light v. Since the wave is a plane wave to the direction $\boldsymbol{\beta}$ of travel, the planes over which the phase is constant (constant phase planes) are perpendicular to the direction $\boldsymbol{\beta}$ of wave travel. This is illustrated graphically in Figure 4-6. To maintain a constant phase (or to keep in step with a constant phase plane), a velocity equal to the speed of light must be maintained in the direction $\boldsymbol{\beta}$ of travel. This is referred to as the phase velocity v_{pr} along the direction $\boldsymbol{\beta}$ of travel. Since the energy also is being transported with the same speed, the energy velocity v_{er} in the direction $\boldsymbol{\beta}$ of travel is also equal to the speed of light. Thus

$$v_r = v_{pr} = v_{er} = v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{4-22}$$

where

$v_r =$ wave velocity in the direction of wave travel
$v_{pr} =$ phase velocity in the direction of wave travel
$v_{er} =$ energy (group) velocity in the direction of wave travel
$v =$ speed of light

To keep in step with a constant phase plane of the wave of Figure 4-6, a velocity in the z direction equal to

$$v_{pz} = \frac{v}{\cos\theta_i} = \frac{1}{\sqrt{\mu\varepsilon}\cos\theta_i} \geq v \tag{4-23}$$

must be maintained. This is referred to as the phase velocity v_{pz} in the z direction, and it is greater than the speed of light. Since nothing travels with speeds greater than the speed of light, it must be remembered that this is a hypothetical velocity that must be maintained in order to

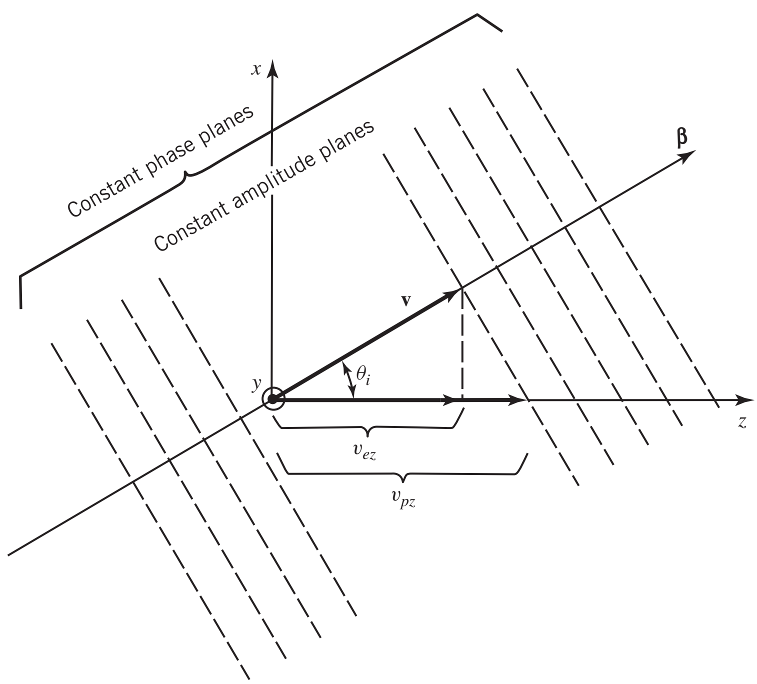


Figure 4-6 Phase and energy (group) velocities of a uniform plane wave.

keep in step with a constant phase plane of the wave that itself travels with the speed of light in the direction $\boldsymbol{\beta}$ of travel. The phase velocities of (4-22) and (4-23) can be obtained, respectively, by using (4-19c) and (4-19d). These are left as end-of-chapter exercises for the reader.

Whereas a velocity greater than the speed of light must be maintained in the z direction to keep in step with a constant phase plane of Figure 4-6, the energy is transported in the z direction with a velocity that is equal to

$$v_{ez} = v\cos\theta_i = \frac{\cos\theta_i}{\sqrt{\mu\varepsilon}} \leq v \tag{4-24}$$

This is referred to as the energy (group) velocity v_{ez} in the z direction, and it is equal to or smaller than the speed of light. Graphically this is illustrated in Figure 4-6.

For any wave, the product of the phase and energy velocities in any direction must be equal to the speed of light squared or

$$v_{pr}v_{er} = v_{pz}v_{ez} = v^2 = \frac{1}{\mu\varepsilon} \tag{4-25}$$

This obviously is satisfied by the previously derived results.

The energy velocity of (4-24) can be derived using (4-18a) and (4-18b) along with the definition (4-9). This is left for the reader as an end-of-chapter exercise.

Since the fields of (4-18a) and (4-18b) form a uniform plane wave, the planes over which the amplitude is maintained constant are also constant planes that are perpendicular to the direction $\boldsymbol{\beta}$ of travel. These are illustrated in Figure 4-6 and coincide with the constant phase planes. For other types of waves, the constant phase and amplitude planes do not in general coincide.

D. Power and Energy Densities The average power density associated with the fields of (4-18a) and (4-18b) that travel in the $\boldsymbol{\beta}^+$ direction is given by

$$\begin{aligned}
\left(\mathbf{S}_{\text{av}}^{+}\right)_r &= \frac{1}{2}\text{Re}\left[\left(\mathbf{E}^{+}\right)\times\left(\mathbf{H}^{+}\right)^{*}\right] \\
&= \frac{1}{2}\text{Re}\left[E_0^{+}\left(\hat{\mathbf{a}}_x\cos\theta_i - \hat{\mathbf{a}}_z\sin\theta_i\right)e^{-j\beta(x\sin\theta_i + z\cos\theta_i)}\right. \\
&\qquad\qquad\left.\times\hat{\mathbf{a}}_y\frac{E_0^{+*}}{\eta}e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}\right] \\
\left(\mathbf{S}_{\text{av}}^{+}\right)_r &= \left(\hat{\mathbf{a}}_x\sin\theta_i + \hat{\mathbf{a}}_z\cos\theta_i\right)\frac{|E_0^{+}|^2}{2\eta} = \hat{\mathbf{a}}_r\frac{|E_0^{+}|^2}{2\eta} = \hat{\mathbf{a}}_x\left(S_{\text{av}}^{+}\right)_x + \hat{\mathbf{a}}_z\left(S_{\text{av}}^{+}\right)_z
\end{aligned} \tag{4-26}$$

where

$$\left(S_{\text{av}}^{+}\right)_x = \sin\theta_i\frac{|E_0^{+}|^2}{2\eta} = \sin\theta_i\left(S_{\text{av}}^{+}\right)_r \tag{4-26a}$$

$$\left(S_{\text{av}}^{+}\right)_z = \cos\theta_i\frac{|E_0^{+}|^2}{2\eta} = \cos\theta_i\left(S_{\text{av}}^{+}\right)_r \tag{4-26b}$$

$(S_{\text{av}}^{+})_r$ represents the average power density along the principal $\boldsymbol{\beta}^+$ direction of travel and $(S_{\text{av}}^{+})_x$ and $(S_{\text{av}}^{+})_z$ represent the directional power densities of the wave in the $+x$ and $+z$ directions, respectively. Similar expressions can be derived for the wave that travels along the $\boldsymbol{\beta}^-$ direction.

Example 4-3

For the TM^y fields of Example 4-2, derive expressions for the average power density along the principal $\boldsymbol{\beta}^+$ direction of travel and for the directional power densities along the $+x$ and $+z$ directions.

Solution: Using the electric and magnetic fields of the solution of Example 4-2 and following the procedure used to derive (4-26) through (4-26b), it can be shown that

$$\begin{aligned}
\left(\mathbf{S}_{\mathrm{av}}^+\right)_r &= \frac{1}{2}\operatorname{Re}\left[\left(\mathbf{E}^+\right) \times \left(\mathbf{H}^+\right)^*\right] \\
&= \frac{1}{2}\operatorname{Re}\left[\hat{\mathbf{a}}_y E_0^+ e^{-j\beta(x\cos\theta_i + y\sin\theta_i)}\right. \\
&\qquad \left. \times \frac{\left(E_0^+\right)^*}{\eta}\left(-\hat{\mathbf{a}}_x\cos\theta_i + \hat{\mathbf{a}}_z\sin\theta_i\right) e^{+j\beta(x\cos\theta_i + y\sin\theta_i)}\right] \\
\left(\mathbf{S}_{\mathrm{av}}^+\right)_r &= \left(\hat{\mathbf{a}}_x\sin\theta_i + \hat{\mathbf{a}}_z\cos\theta_i\right)\frac{|E_0^+|^2}{2\eta} = \hat{\mathbf{a}}_r\frac{|E_0^+|^2}{2\eta} \\
&= \hat{\mathbf{a}}_x\left(S_{\mathrm{av}}^+\right)_x + \hat{\mathbf{a}}_z\left(S_{\mathrm{av}}^+\right)_z
\end{aligned}$$

where

$$\begin{aligned}
\left(S_{\mathrm{av}}^+\right)_x &= \sin\theta_i\frac{|E_0^+|^2}{2\eta} = \sin\theta_i\left(S_{\mathrm{av}}^+\right)_r \\
\left(S_{\mathrm{av}}^+\right)_z &= \cos\theta_i\frac{|E_0^+|^2}{2\eta} = \cos\theta_i\left(S_{\mathrm{av}}^+\right)_r
\end{aligned}$$

(S_{av}^+), $(S_{\mathrm{av}}^+)_x$, and $(S_{\mathrm{av}}^+)_z$ of this TM^y wave are identical to the corresponding ones of the TE^y wave, given by (4-26) through (4-26b).

4.3 TRANSVERSE ELECTROMAGNETIC MODES IN LOSSY MEDIA

In addition to the accumulation of phase, electromagnetic waves that travel in lossy media undergo attenuation. To account for the attenuation, an attenuation constant is introduced as discussed in Chapter 3, Section 3.4.1B. In this section we want to discuss the solution for the electric and magnetic fields of uniform plane waves as they travel in lossy media [5].

4.3.1 Uniform Plane Waves in an Unbounded Lossy Medium — Principal Axis

As for the electromagnetic wave of Section 4.2.1, let us assume that a uniform plane wave is traveling in a lossy medium. Using the coordinate system of Figure 4-1, the electric field is assumed to have an x component and the wave is traveling in the $\pm z$ direction. Since the electric field must satisfy the wave equation for lossy media, its expression takes, according to (3-42a), the form

$$\mathbf{E}(z) = \hat{\mathbf{a}}_x E_x(z) = \hat{\mathbf{a}}_x\left(E_0^+ e^{-\gamma z} + E_0^- e^{+\gamma z}\right) = \hat{\mathbf{a}}_x\left(E_0^+ e^{-\alpha z} e^{-j\beta z} + E_0^- e^{+\alpha z} e^{+j\beta z}\right) \tag{4-27}$$

where $\gamma_x = \gamma_y = 0$ and $\gamma_z = \gamma$. The first term represents the positive traveling wave and the second term represents the negative traveling wave. In (4-27) γ is the propagation constant whose real α and imaginary β parts are defined, respectively, as the attenuation and phase constants. According to (3-37e) and (3-46), γ takes the form

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \sqrt{-\omega^2\mu\varepsilon + j\omega\mu\sigma} \tag{4-28}$$

Squaring (4-28) and equating real and imaginary from both sides reduces it to

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon \tag{4-28a}$$

$$2\alpha\beta = \omega\mu\sigma \tag{4-28b}$$

Solving (4-28a) and (4-28b) simultaneously, we can write α and β as

$$\boxed{\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]\right\}^{1/2} \text{ Np/m}} \tag{4-28c}$$

$$\boxed{\beta = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]\right\}^{1/2} \text{ rad/m}} \tag{4-28d}$$

In the literature, the phase constant β is also represented by k.

The attenuation constant α is often expressed in decibels per meter (dB/m). The conversion between Nepers per meter and decibels per meter is obtained by examining the real exponential in (4-27) that represents the attenuation factor of the wave in a lossy medium. Since that factor represents the relative attenuation of the electric or magnetic field, its conversion to decibels (dB) is obtained by

$$\begin{aligned} \text{dB} &= 20\log_{10}\left(e^{-\alpha z}\right) = 20(-\alpha z)\log_{10}(e) \\ &= 20(-\alpha z)(0.434) = -8.68(\alpha z) \end{aligned} \tag{4-28e}$$

or

$$|\alpha\,(\text{Np/m})| = \frac{1}{8.68}|\alpha\,(\text{dB/m})| \tag{4-28f}$$

The magnetic field associated with the electric field of (4-27) can be obtained using Maxwell's equation 4-3 or 4-3a, that is,

$$\mathbf{H} = -\frac{1}{j\omega\mu}\nabla\times\mathbf{E} = -\hat{\mathbf{a}}_y\frac{1}{j\omega\mu}\frac{\partial E_x}{\partial z} \tag{4-29}$$

Using (4-27) reduces (4-29) to

$$\begin{aligned} \mathbf{H} &= +\hat{\mathbf{a}}_y\frac{\gamma}{j\omega\mu}\left(E_0^+e^{-\gamma z} - E_0^-e^{+\gamma z}\right) \\ &= \hat{\mathbf{a}}_y\frac{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}}{j\omega\mu}\left(E_0^+e^{-\gamma z} - E_0^-e^{+\gamma z}\right) \\ &= \hat{\mathbf{a}}_y\sqrt{\frac{\sigma + j\omega\varepsilon}{j\omega\mu}}\left(E_0^+e^{-\gamma z} - E_0^-e^{+\gamma z}\right) \end{aligned}$$

$$\mathbf{H} = \hat{\mathbf{a}}_y\frac{1}{Z_w}\left(E_0^+e^{-\gamma z} - E_0^-e^{+yz}\right) \tag{4-29a}$$

In (4-29a), Z_w is the wave impedance of the wave, and it takes the form

$$Z_w = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \eta_c \tag{4-30}$$

which is also equal to the intrinsic impedance η_c of the lossy medium. The equality between the wave and intrinsic impedances for TEM waves in lossy media is identical to that for lossless media of Section 4.2.1B.

The average power density associated with the positive traveling fields of (4-27) and (4-29a) can be written as

$$\mathbf{S}^+ = \frac{1}{2}\mathrm{Re}\left(\mathbf{E}^+ \times \mathbf{H}^{+*}\right) = \frac{1}{2}\mathrm{Re}\left(\hat{\mathbf{a}}_x E_0^+ e^{-\alpha z} e^{-j\beta z} \times \hat{\mathbf{a}}_y \frac{E_0^{+*}}{\eta_c^*} e^{-\alpha z} e^{+j\beta z}\right)$$

$$\mathbf{S}^+ = \hat{\mathbf{a}}_z \frac{|E_0^+|^2}{2} e^{-2\alpha z}\mathrm{Re}\left[\frac{1}{\eta_c^*}\right] \tag{4-31}$$

Individually each term of (4-27) or (4-29a) represents a traveling wave in its respective direction. The magnitude of each term in (4-27) takes the form

$$|E_x^+(z)| = |E_0^+|e^{-\alpha z} \tag{4-32a}$$

$$|E_x^-(z)| = |E_0^-|e^{+\alpha z} \tag{4-32b}$$

which, when plotted for $-\lambda \le z \le +\lambda$ and $|\Gamma| = 0.2$ through 1 (in increments of 0.2), take the form shown in Figure 4-7*a*.

Collectively, both terms in each of the fields in (4-27) or (4-29a) represent a standing wave. Using the procedure outlined in Section 4.2.1D, (4-27) can also be written as

$$E_x(z) = \sqrt{\left(E_0^+\right)^2 e^{-2\alpha z} + \left(E_0^-\right)^2 e^{+2\alpha z} + 2E_0^+E_0^- \cos(2\beta z)} \times \exp\left\{-j\tan^{-1}\left[\frac{E_0^+ e^{-\alpha z} - E_0^- e^{+\alpha z}}{E_0^+ e^{-\alpha z} + E_0^- e^{+\alpha z}}\tan(\beta z)\right]\right\} \tag{4-33}$$

The standing wave pattern is given by the amplitude term of

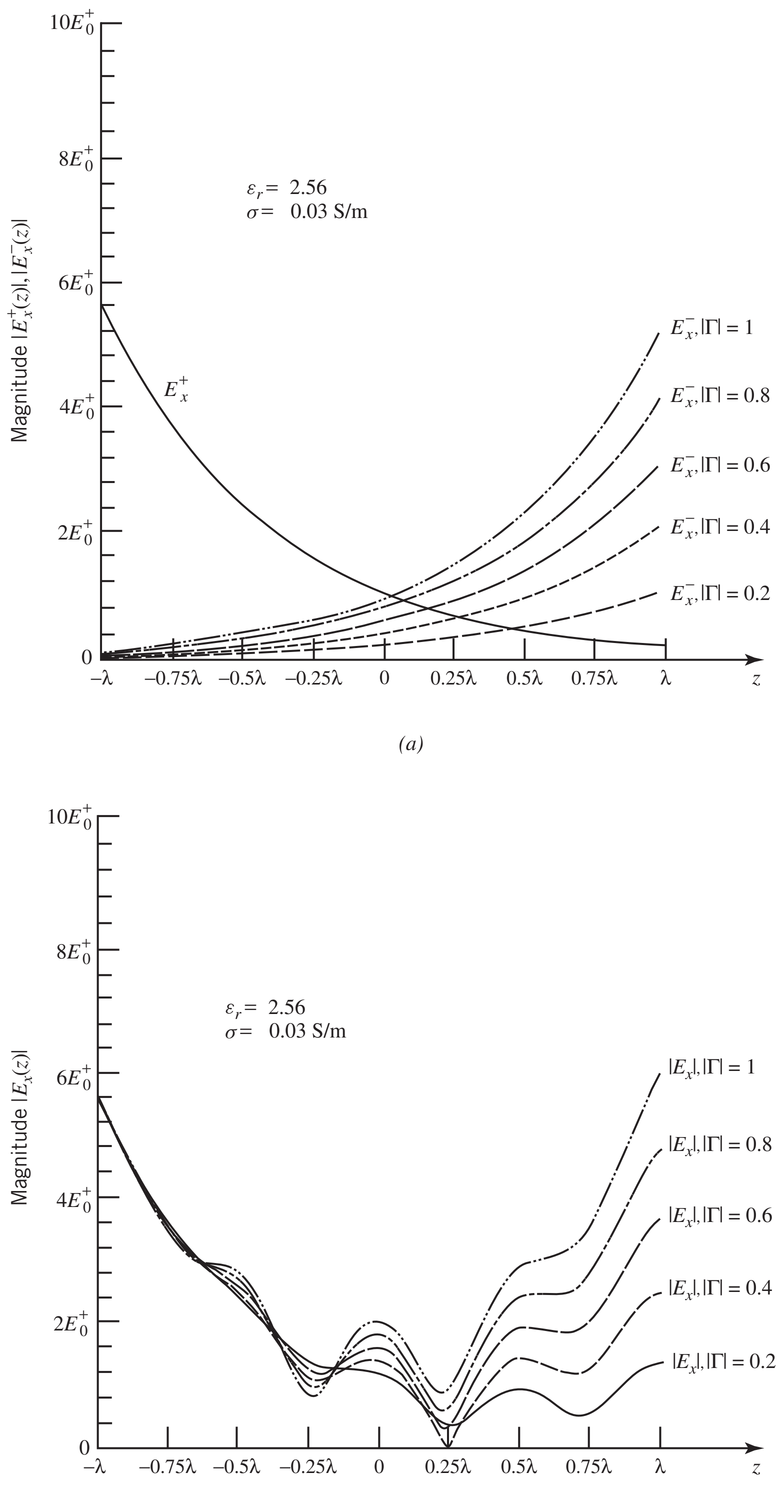
$$|E_x(z)| = \sqrt{\left(E_0^+\right)^2 e^{-2\alpha z} + \left(E_0^-\right)^2 e^{+2\alpha z} + 2E_0^+E_0^- \cos(2\beta z)} \tag{4-33a}$$

which for $|\Gamma| = E_0^-/E_0^+ = 0.2$ through 1, in increments of 0.2, is shown plotted in Figure 4-7*b* in the range $-\lambda \le z \le \lambda$ when $f = 100$ MHz, $\varepsilon_r = 2.56$, $\mu_r = 1$, and $\sigma = 0.03$ S/m.

The distance the wave must travel in a lossy medium to reduce its value to $e^{-1} = 0.368 = 36.8\%$ *is defined as the skin depth* δ. For each of the terms of (4-27) or (4-29a), this distance is

$$\boxed{\delta = \text{skin depth} = \frac{1}{\alpha} = \frac{1}{\omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1 + (\sigma/\omega\varepsilon)^2} - 1\right]\right\}^{1/2}}\ \text{m}} \tag{4-34}$$

In summary, the attenuation constant α, phase constant β, wave Z_w and intrinsic η_c impedances, wavelength λ, velocity v, and skin depth δ for a uniform plane wave traveling in a lossy medium are listed in the second column of Table 4-1. The same expressions are valid for plane and TEM waves. Simpler expressions for each can be derived depending upon the value of the $(\sigma/\omega\varepsilon)^2$ ratio. Media whose $(\sigma/\omega\varepsilon)^2$ is much less than unity $[(\sigma/\omega\varepsilon)^2 \ll 1]$ are referred to as *good dielectrics* and those whose $(\sigma/\omega\varepsilon)^2$ is much greater than unity $[(\sigma/\omega\varepsilon)^2 \gg 1]$ are referred to as *good conductors* [6]; each will now be discussed.



(a)

(b)

Figure 4-7 Wave patterns of uniform plane waves in a lossy medium. (*a*) Traveling. (*b*) Standing.

TABLE 4-1 Propagation constant, wave impedance, wavelength, velocity, and skin depth of TEM wave in lossy media

	Exact	Good dielectric $\left(\frac{\sigma}{\omega\varepsilon}\right)^2 \ll 1$	Good conductor $\left(\frac{\sigma}{\omega\varepsilon}\right)^2 \gg 1$
Attenuation constant α	$= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]\right\}^{1/2}$	$\simeq \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\simeq \sqrt{\frac{\omega\mu\sigma}{2}}$
Phase constant β	$= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]\right\}^{1/2}$	$\simeq \omega\sqrt{\mu\varepsilon}$	$\simeq \sqrt{\frac{\omega\mu\sigma}{2}}$
Wave Z_w intrinsic η_c impedances $Z_w = \eta_c$	$= \sqrt{\frac{\mathrm{j}\omega\mu}{\sigma + \mathrm{j}\omega\varepsilon}}$	$\simeq \sqrt{\frac{\mu}{\varepsilon}}$	$\simeq \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$
Wavelength λ	$= \frac{2\pi}{\beta}$	$\simeq \frac{2\pi}{\omega\sqrt{\mu\varepsilon}}$	$\simeq 2\pi\sqrt{\frac{2}{\omega\mu\sigma}}$
Velocity v	$= \frac{\omega}{\beta}$	$\simeq \frac{1}{\sqrt{\mu\varepsilon}}$	$\simeq \sqrt{\frac{2\omega}{\mu\sigma}}$
Skin depth δ	$= \frac{1}{\alpha}$	$\simeq \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$	$\simeq \sqrt{\frac{2}{\omega\mu\sigma}}$

***A. Good Dielectrics* $[(\sigma/\omega\varepsilon)^2 \ll 1]$** For source-free lossy media, Maxwell's equation in differential form as derived from Ampere's law takes the form, by referring to Table 1-4, of

$$\nabla \times \mathbf{H} = \mathbf{J}_c + \mathbf{J}_d = \sigma\mathbf{E} + j\omega\varepsilon\mathbf{E} = (\sigma + j\omega\varepsilon)\,\mathbf{E} \tag{4-35}$$

where $\mathbf{J}_c$ and $\mathbf{J}_d$ represent, respectively, the conduction and displacement current densities. When $\sigma/\omega\varepsilon \ll 1$, the displacement current density is much greater than the conduction current density; when $\sigma/\omega\varepsilon \gg 1$ the conduction current density is much greater than the displacement current density. For each of these two cases, the exact forms of the field parameters of Table 4-1 can be approximated by simpler forms. This will be demonstrated next.

For a good dielectric [when $(\sigma/\omega\varepsilon)^2 \ll 1$], the exact expression for the attenuation constant of (4-28c) can be written using the binomial expansion and it takes the form

$$\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]\right\}^{1/2}$$

$$\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\left(1+\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2 - \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^4 \cdots\right) - 1\right]\right\}^{1/2} \tag{4-36}$$

Retaining only the first two terms of the infinite series, (4-36) can be approximated by

$$\alpha \simeq \omega\sqrt{\mu\varepsilon}\left[\frac{1}{4}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \tag{4-36a}$$

In a similar manner it can be shown that by following the same procedure but only retaining the first term of the infinite series, the exact expression for β of (4-28d) can be approximated by

$$\beta \simeq \omega\sqrt{\mu\varepsilon} \tag{4-37}$$

For good dielectrics, the wave and intrinsic impedances of (4-30) can be approximated by

$$Z_w = \eta_c = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu/j\omega\varepsilon}{\sigma/j\omega\varepsilon + 1}} \simeq \sqrt{\frac{\mu}{\varepsilon}} \tag{4-38}$$

while the skin depth can be represented by

$$\delta = \frac{1}{\alpha} \simeq \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}} \tag{4-39}$$

These and other approximate forms for the parameters of good dielectrics are summarized on the third column of Table 4-1.

***B. Good Conductors* [$(\sigma/\omega\varepsilon)^2 \gg 1$]** For good conductors, the exact expression for the attenuation constant of (4-28c) can be written using the binomial expansion and takes the form

$$\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{\left(\frac{\sigma}{\omega\varepsilon}\right)^2 + 1} - 1\right]\right\}^{1/2} = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\frac{\sigma}{\omega\varepsilon}\left(1 + \frac{1}{(\sigma/\omega\varepsilon)^2}\right)^{1/2} - 1\right]\right\}^{1/2}$$

$$\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\frac{\sigma}{\omega\varepsilon} + \frac{1}{2}\frac{1}{\sigma/\omega\varepsilon} - \frac{1}{8}\frac{1}{(\sigma/\omega\varepsilon)^3} + \cdots - 1\right]\right\}^{1/2} \tag{4-40}$$

Retaining only the first term of the infinite series expansion, (4-40) can be approximated by

$$\alpha \simeq \omega\sqrt{\mu\varepsilon}\left(\frac{1}{2}\frac{\sigma}{\omega\varepsilon}\right)^{1/2} = \sqrt{\frac{\omega\mu\sigma}{2}} \tag{4-40a}$$

Following a similar procedure, the phase constant of (4-28d) can be approximated by

$$\beta \simeq \sqrt{\frac{\omega\mu\sigma}{2}} \tag{4-41}$$

which is identical to the approximate expression for the attenuation constant of (4-40a).

For good conductors, the wave and intrinsic impedances of (4-30) can be approximated by

$$Z_w = \eta_c = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu/\omega\varepsilon}{\sigma/\omega\varepsilon + j}} \simeq \sqrt{j\frac{\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{2\sigma}}(1 + j) \tag{4-42}$$

whose real and imaginary parts are identical. For the same conditions, the skin depth can be approximated by

$$\delta = \frac{1}{\alpha} \simeq \sqrt{\frac{2}{\omega\mu\sigma}} \tag{4-43}$$

This is the most widely recognized form for the skin depth.

4.3.2 Uniform Plane Waves in an Unbounded Lossy Medium — Oblique Angle

For lossy media the difference between principal axes propagation and propagation at oblique angles is that the propagation constant γ_r along the direction $\boldsymbol{\beta}$ of propagation must be decomposed into its directional components along the principal axes of the coordinate system. In addition, since the propagation constant γ has real (α) and imaginary (β) parts, constant amplitude and constant phase planes are associated with the wave. As discussed in Section 4.2.2C and illustrated in Figure 4-6, the constant phase planes for a uniform plane wave are planes that are parallel to each other, perpendicular to the direction of propagation, and coincide with the constant amplitude planes. The constant amplitude planes are planes over which the amplitude remains constant. For a uniform plane wave traveling in a lossy medium, the constant amplitude planes are also parallel to each other, are perpendicular to the direction of travel, and coincide with the constant phase planes. This is illustrated in Figure 4-6 for a uniform plane wave traveling at an oblique angle in a lossless medium.

Let us assume that a uniform plane wave that is also TE^y is traveling in a lossy medium at an angle θ_i, as shown in Figure 4-4a. Following a procedure similar to the lossless case and referring to (4-17a) and (4-17b), the propagation constant of (4-28) can now be written for the positive and negative traveling waves as

$$\boldsymbol{\gamma}^+ = \gamma\,(\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i) = (\alpha + j\beta)\,(\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i) \tag{4-44a}$$

$$\boldsymbol{\gamma}^- = -\gamma\,(\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i) = -\,(\alpha + j\beta)\,(\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i) \tag{4-44b}$$

where the real (α) and imaginary (β) parts of γ are given by (4-28c) and (4-28d), respectively. Using (4-44a) and (4-44b), the electric and magnetic fields can be written, by referring to (4-17) through (4-18c), as

$$\begin{aligned}
\mathbf{E} &= E_0^+\,(\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\,e^{-\boldsymbol{\gamma}^+\cdot\mathbf{r}} + E_0^-\,(\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\,e^{-\boldsymbol{\gamma}^-\cdot\mathbf{r}} \\
\mathbf{E} &= E_0^+\,(\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\,e^{-(\alpha+j\beta)(x\sin\theta_i + z\cos\theta_i)} \\
&\quad + E_0^-\,(\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\,e^{+(\alpha+j\beta)(x\sin\theta_i + z\cos\theta_i)}
\end{aligned} \tag{4-45a}$$

$$\mathbf{H} = \hat{\mathbf{a}}_y \left[\frac{E_0^+}{\eta_c} e^{-(\alpha+j\beta)(x\sin\theta_i + z\cos\theta_i)} - \frac{E_0^-}{\eta_c} e^{+(\alpha+j\beta)(x\sin\theta_i + z\cos\theta_i)}\right] \tag{4-45b}$$

Because the wave is a uniform plane wave in the $\boldsymbol{\beta}$ direction of propagation, the wave impedance Z_{wr} in the direction of propagation is equal to the intrinsic impedance η_c of the lossy medium given by (4-30) or

$$Z_{wr} = \eta_c = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \tag{4-46}$$

However, the directional impedances in the x and z directions are given, by referring to (4-20a) and (4-20b), by

$$Z_x^+ = -\frac{E_z^+}{H_y^+} = \eta_c \sin\theta_i = Z_x^- = \frac{E_z^-}{H_y^-} \tag{4-47a}$$

$$Z_z^+ = \frac{E_x^+}{H_y^+} = \eta_c \cos\theta_i = Z_z^- = -\frac{E_x^-}{H_y^-} \tag{4-47b}$$

According to (4-22) through (4-24) the phase and energy velocities in the principal $\boldsymbol{\beta}$ direction of travel and in the z direction are given, respectively, by

$$v_r = v_{pr} = v_{er} = v = \frac{\omega}{\beta} \tag{4-48a}$$

$$v_{pz} = \frac{v}{\cos\theta_i} = \frac{\omega}{\beta\cos\theta_i} \geq v = \frac{\omega}{\beta} \tag{4-48b}$$

$$v_{ez} = v\cos\theta_i = \frac{\omega}{\beta}\cos\theta_i \leq v = \frac{\omega}{\beta} \tag{4-48c}$$

where β for a lossy medium is given by (4-28d) or

$$\beta = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]\right\}^{1/2} \tag{4-48d}$$

As for the lossless medium, the product of the phase and energy velocities is equal to the square of the velocity of light v in the lossy medium, or

$$v_{pr} v_{er} = v_{pz} v_{ez} = v^2 \tag{4-48e}$$

Using the procedure followed to derive (4-26) through (4-26b) and (4-31), the average power density along the principal direction $\boldsymbol{\beta}$ of travel and the directional power densities along the x and z directions can be written for the fields of (4-45a) and (4-45b) as

$$\begin{aligned}\left(\mathbf{S}_{\mathrm{av}}^{+}\right)_r &= (\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i)\frac{\left|E_0^+\right|^2}{2}e^{-2\alpha(x\sin\theta_i+z\cos\theta_i)}\mathrm{Re}\left[\frac{1}{\eta_c^*}\right]\\ &= \hat{\mathbf{a}}_r\frac{\left|E_0^+\right|^2}{2}e^{-2\alpha r}\mathrm{Re}\left[\frac{1}{\eta_c^*}\right]\end{aligned} \tag{4-49a}$$

$$\begin{aligned}\left(S_{\mathrm{av}}^{+}\right)_x &= \sin\theta_i\frac{\left|E_0^+\right|^2}{2}e^{-2\alpha(x\sin\theta_i+z\cos\theta_i)}\mathrm{Re}\left[\frac{1}{\eta_c^*}\right]\\ &= \sin\theta_i\frac{\left|E_0^+\right|^2}{2}e^{-2\alpha r}\mathrm{Re}\left[\frac{1}{\eta_c^*}\right]\end{aligned} \tag{4-49b}$$

$$\begin{aligned}\left(S_{\mathrm{av}}^{+}\right)_z &= \cos\theta_i\frac{\left|E_0^+\right|^2}{2}e^{-2\alpha(x\sin\theta_i+z\cos\theta_i)}\mathrm{Re}\left[\frac{1}{\eta_c^*}\right]\\ &= \cos\theta_i\frac{\left|E_0^+\right|^2}{2}e^{-2\alpha r}\mathrm{Re}\left[\frac{1}{\eta_c^*}\right]\end{aligned} \tag{4-49c}$$

Example 4-4

For a TM^y wave traveling in a lossy medium at an oblique angle θ_i, derive expressions for the fields, wave impedances, phase and energy velocities, and average power densities.

Solution: The solution to this problem can be accomplished by following the procedure used to derive the expressions of the fields and other wave characteristics of a TE^y wave traveling at an oblique angle in a lossy medium, as outlined in this section, and referring to the solution of Examples 4-2 and 4-3. Doing this we can write the fields of a TM^y traveling in a lossy medium at an oblique angle θ_i, the coordinate system of which is illustrated in Figure 4-4*b*, as

$$\mathbf{E} = \mathbf{E}^+ + \mathbf{E}^- = \hat{\mathbf{a}}_y\left[E_0^+ e^{-(\alpha+j\beta)(x\sin\theta_i+z\cos\theta_i)} + E_0^- e^{+(\alpha+j\beta)(x\sin\theta_i+z\cos\theta_i)}\right]$$

$$\begin{aligned}\mathbf{H} = \mathbf{H}^+ + \mathbf{H}^- = {}& \frac{E_0^+}{\eta_c}(-\hat{\mathbf{a}}_x\cos\theta_i + \hat{\mathbf{a}}_z\sin\theta_i)\,e^{-(\alpha+j\beta)(x\sin\theta_i+z\cos\theta_i)}\\ &+\frac{E_0^-}{\eta_c}(\hat{\mathbf{a}}_x\cos\theta_i - \hat{\mathbf{a}}_z\sin\theta_i)\,e^{+(\alpha+j\beta)(x\sin\theta_i+z\cos\theta_i)}\end{aligned}$$

In addition, the wave impedances are given, by referring to (4-21a) and (4-21b), by

$$Z_x^+ = \frac{E_y^+}{H_z^+} = \frac{\eta_c}{\sin\theta_i} = Z_x^- = -\frac{E_y^-}{H_z^-}$$

$$Z_z^+ = -\frac{E_y^+}{H_x^+} = \frac{\eta_c}{\cos\theta_i} = Z_z^- = \frac{E_y^-}{H_x^-}$$

The phase and energy velocities, and their relationships, are the same as those for the TE^y wave, as given by (4-48a) through (4-48e). Similarly, the average power densities are those given by (4-49a) through (4-49c).

4.4 POLARIZATION

According to the *IEEE Standard Definitions for Antennas* [7, 8], the *polarization of a radiated wave* is defined as "that property of a radiated electromagnetic wave describing the time-varying direction and relative magnitude of the electric field vector; specifically, the figure traced as a function of time by the extremity of the vector at a fixed location in space, and the sense in which it is traced, as observed along the direction of propagation." In other words, polarization is the curve traced out, at a given observation point as a function of time, by the end point of the arrow representing the instantaneous electric field. The field must be observed along the direction of propagation. A typical trace as a function of time is shown in Figure 4-8 [8].

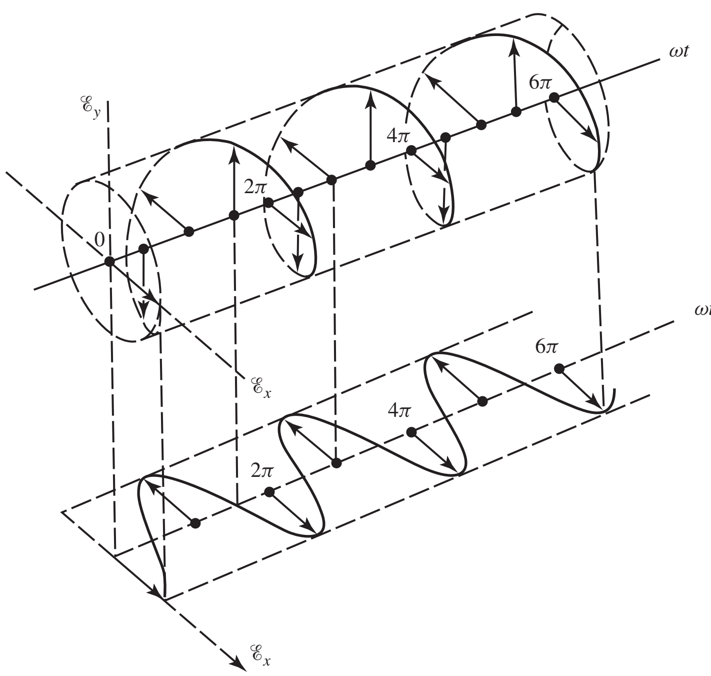


Figure 4-8 Rotation of a plane electromagnetic wave at $z = 0$ as a function of time. (Source: C. A. Balanis, *Antenna Theory: Analysis and Design*. 3rd Edition. Copyright © 2005, John Wiley & Sons, Inc. Reprinted by permission of John Wiley & Sons, Inc.).

Polarization may be classified into three categories: *linear, circular, and elliptical* [8]. If the vector that describes the electric field at a point in space as a function of time is always directed along a line, which is normal to the direction of propagation, the field is said to be *linearly* polarized. In general, however, the figure that the electric field traces is an ellipse, and the field is said to be *elliptically* polarized. Linear and circular polarizations are special cases of elliptical, and they can be obtained when the ellipse becomes a straight line or a circle, respectively. The figure of the electric field is traced in a *clockwise* (CW) or *counterclockwise* (CCW) sense. Clockwise rotation of the electric field vector is also designated as *right-hand polarization* and counterclockwise as *left-hand polarization*. In Figure 4-9 we show the figure traced by the extremity of the time-varying field vector for linear, circular, and elliptical polarizations.

The mathematical details for defining linear, circular, and elliptical polarizations follow.

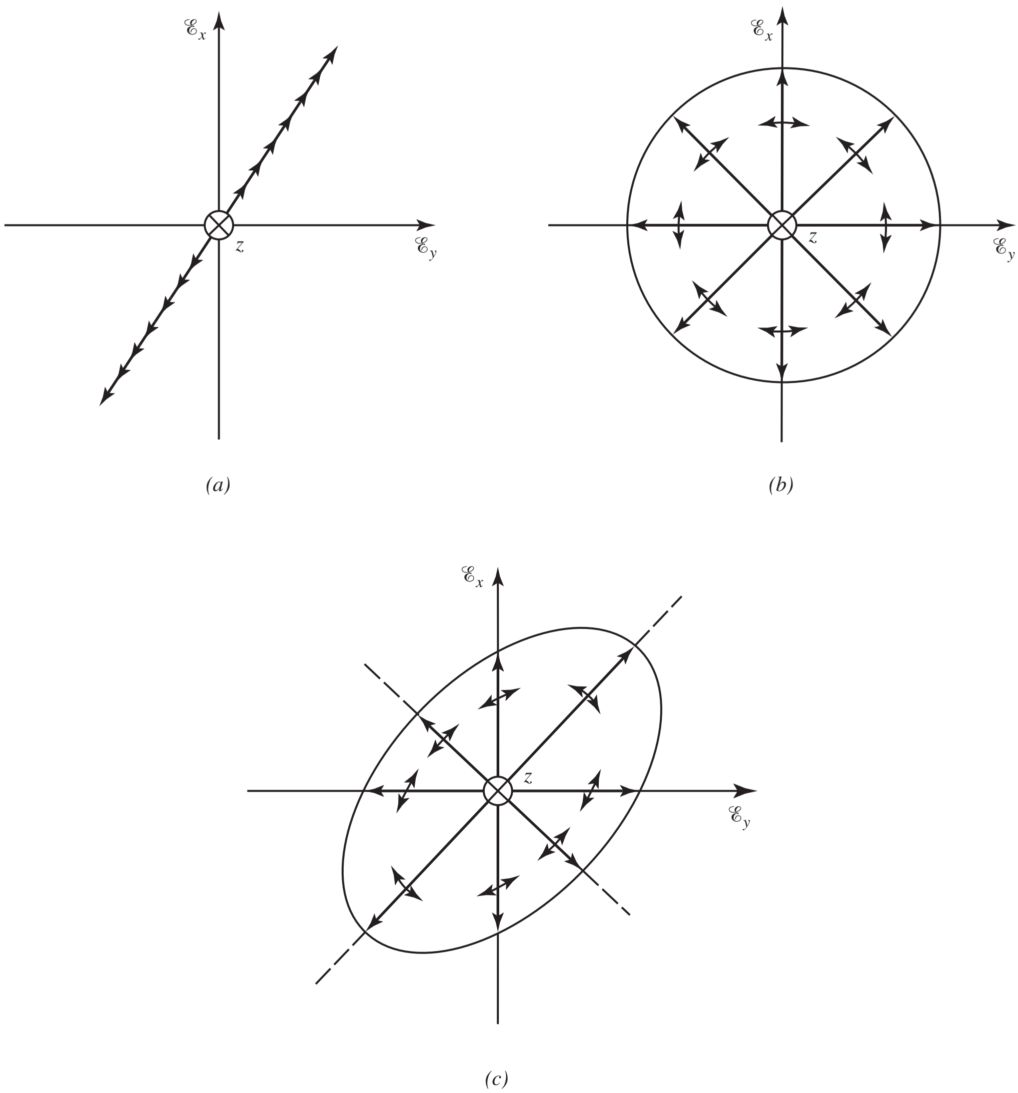


Figure 4-9 Polarization figure traces of an electric field extremity *as a function of time for a fixed position*. (*a*) Linear. (*b*) Circular. (*c*) Elliptical.

4.4.1 Linear Polarization

Let us consider a harmonic plane wave, with x and y electric field components, traveling in the positive z direction (into the page), as shown in Figure 4-10 [8]. The instantaneous electric and magnetic fields are given by

$$\begin{aligned}\boldsymbol{\mathscr{E}} &= \hat{\mathbf{a}}_x \mathscr{E}_x + \hat{\mathbf{a}}_y \mathscr{E}_y = \operatorname{Re}\left[\hat{\mathbf{a}}_x E_x^+ e^{j(\omega t - \beta z)} + \hat{\mathbf{a}}_y E_y^+ e^{j(\omega t-\beta z)}\right] \\ &= \hat{\mathbf{a}}_x E_{x_0}^+ \cos(\omega t - \beta z + \phi_x) + \hat{\mathbf{a}}_y E_{y_0}^+ \cos\left(\omega t - \beta z + \phi_y\right)\end{aligned} \tag{4-50a}$$

$$\begin{aligned}\boldsymbol{\mathscr{H}} &= \hat{\mathbf{a}}_y \mathscr{H}_y + \hat{\mathbf{a}}_x \mathscr{H}_x = \operatorname{Re}\left[\hat{\mathbf{a}}_y \frac{E_x^+}{\eta} e^{j(\omega t - \beta z)} - \hat{\mathbf{a}}_x \frac{E_y^+}{\eta} e^{j(\omega t-\beta z)}\right] \\ &= \hat{\mathbf{a}}_y \frac{E_{x_0}^+}{\eta} \cos(\omega t - \beta z + \phi_x) - \hat{\mathbf{a}}_x \frac{E_{y_0}^+}{\eta} \cos\left(\omega t - \beta z + \phi_y\right)\end{aligned} \tag{4-50b}$$

where E_x^+, E_y^+ are complex and $E_{x_0}^+$, $E_{y_0}^+$ are real.

Let us now examine the variation of the instantaneous electric field vector $\mathscr{E}$ as given by (4-50a) at the $z = 0$ plane. Other planes may be considered, but the $z = 0$ plane is chosen for convenience and simplicity. For the first example, let

$$E_{y_0}^+ = 0 \tag{4-51}$$

in (4-50a). Then

$$\begin{aligned}\mathscr{E}_x &= E_{x_0}^+ \cos(\omega t + \phi_x) \\ \mathscr{E}_y &= 0\end{aligned} \tag{4-51a}$$

The locus of the instantaneous electric field vector is given by

$$\boldsymbol{\mathscr{E}} = \hat{\mathbf{a}}_x E_{x_0}^+ \cos(\omega t + \phi_x) \tag{4-51b}$$

which is a straight line, and it will always be directed along the x axis at all times, as shown in Figure 4-10. The field is said to be *linearly polarized in the x direction*.

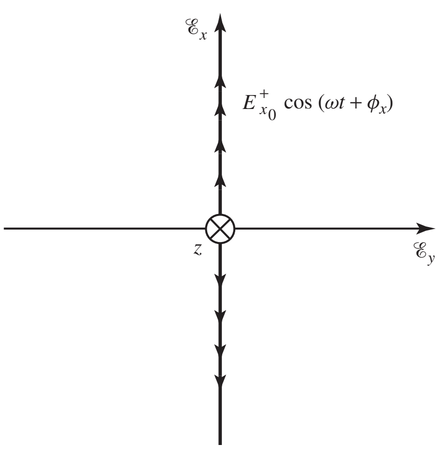


Figure 4-10 Linearly polarized field in the x direction.

Example 4-5

Determine the polarization of the wave given by (4-50a) when $E_{x0}^{+} = 0$.

Solution: Since

$$E_{x0}^{+} = 0$$

then

$$\mathscr{E}_x = 0$$
$$\mathscr{E}_y = E_{y0}^{+} \cos\left(\omega t + \phi_y\right)$$

The locus of the instantaneous electric field vector is given by

$$\boldsymbol{\mathscr{E}} = \hat{\mathbf{a}}_y E_{y0}^{+} \cos\left(\omega t + \phi_y\right)$$

which again is a straight line but directed along the y axis at all times, as shown in Figure 4-11. The field is said to be *linearly polarized in the y direction*.

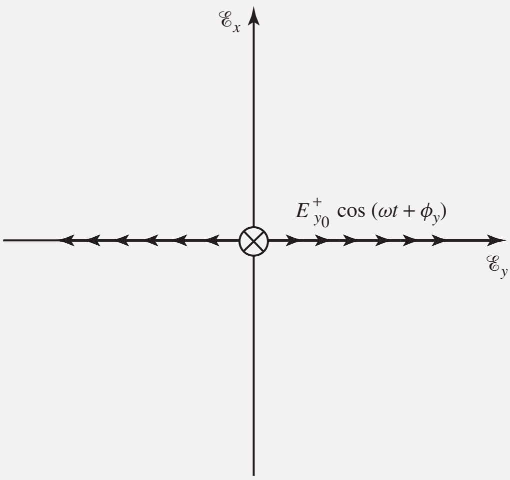


Figure 4-11 Linearly polarized field in the y direction.

Example 4-6

Determine the polarization and direction of polarization of the wave given by (4-50a) when $\phi_x = \phi_y = \phi$.

Solution: Since

$$\phi_x = \phi_y = \phi$$

then

$$\mathscr{E}_x = E_{x0}^{+} \cos(\omega t + \phi)$$
$$\mathscr{E}_y = E_{y0}^{+} \cos(\omega t + \phi)$$

The amplitude of the electric field vector is given by

$$\mathscr{E} = \sqrt{\mathscr{E}_x^2 + \mathscr{E}_y^2} = \sqrt{\left(E_{x0}^+\right)^2 + \left(E_{y0}^+\right)^2}\cos(\omega t + \phi)$$

which is a straight line directed at all times along a line that makes an angle ψ with the x axis as shown in Figure 4-12. The angle ψ is given by

$$\psi = \tan^{-1}\left[\frac{\mathscr{E}_y}{\mathscr{E}_x}\right] = \tan^{-1}\left[\frac{E_{y0}^+}{E_{x0}^+}\right]$$

The field is said to be *linearly polarized in the* ψ *direction*.

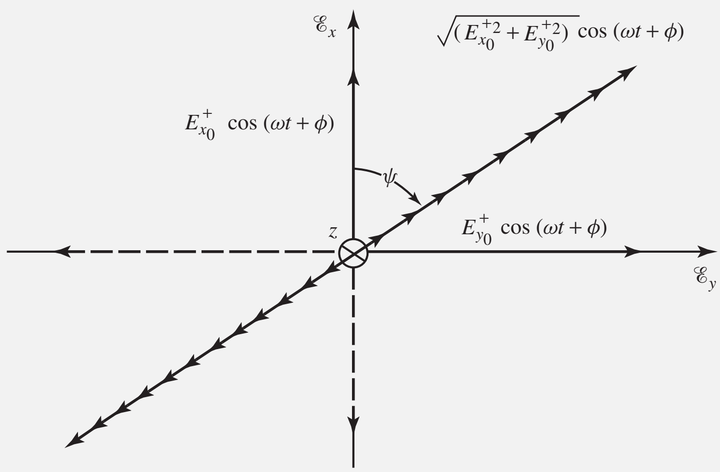


Figure 4-12 Linearly polarized field in the ψ direction.

It is evident from the preceding examples that *a time-harmonic field is linearly polarized at a given point in space if the electric field* (*or magnetic field*) *vector at that point is oriented along the same straight line at every instant of time*. This is accomplished if the field vector (electric or magnetic) possesses (a) only one component or (b) two orthogonal linearly polarized components that are in time phase or integer multiples of 180° out of phase.

4.4.2 Circular Polarization

A wave is said to be *circularly polarized if the tip of the electric field vector traces out a circular locus in space*. At various instants of time, the electric field intensity of such a wave always has the same amplitude and the orientation in space of the electric field vector changes continuously with time in such a manner as to describe a circular locus [8, 9].

A. Right-Hand (Clockwise) Circular Polarization A wave has *right-hand circular polarization* if its electric field vector has a clockwise sense of rotation *when it is viewed along the axis of propagation*. In addition, the electric field vector must trace a circular locus if the wave is to have also a circular polarization.

Let us examine the locus of the instantaneous electric field vector ($\boldsymbol{\mathscr{E}}$) at the $z = 0$ plane at all times. For this particular example, let in (4-50a)

$$\phi_x = 0$$
$$\phi_y = -\pi/2$$
$$E_{x_0}^+ = E_{y_0}^+ = E_{\mathrm{R}} \tag{4-52}$$

Then

$$\mathscr{E}_x = E_{\mathrm{R}} \cos(\omega t)$$
$$\mathscr{E}_y = E_{\mathrm{R}} \cos\left(\omega t - \frac{\pi}{2}\right) = E_{\mathrm{R}} \sin(\omega t) \tag{4-52a}$$

The locus of the amplitude of the electric field vector is given by

$$\mathscr{E} = \sqrt{\mathscr{E}_x^2 + \mathscr{E}_y^2} = \sqrt{E_{\mathrm{R}}^2(\cos^2 \omega t + \sin^2 \omega t)} = E_{\mathrm{R}} \tag{4-52b}$$

and it is directed along a line making an angle ψ with the x axis, which is given by

$$\psi = \tan^{-1}\left[\frac{\mathscr{E}_y}{\mathscr{E}_x}\right] = \tan^{-1}\left[\frac{E_{\mathrm{R}} \sin(\omega t)}{E_{\mathrm{R}} \cos(\omega t)}\right] = \tan^{-1}[\tan(\omega t)] = \omega t \tag{4-52c}$$

If we plot the locus of the electric field vector for various times at the $z = 0$ plane, we see that it forms a circle of radius E_{R} and it rotates clockwise with an angular frequency ω, as shown in Figure 4-13. Thus the wave is said to have a *right-hand circular polarization*. Remember that the rotation is viewed from the "rear" of the wave in the direction of propagation. In this example, the wave is traveling in the positive z direction (into the page) so that the rotation is examined from an observation point looking into the page and perpendicular to it.

We can write the instantaneous electric field vector as

$$\begin{aligned}\boldsymbol{\mathscr{E}} &= \mathrm{Re}\left[\hat{\mathbf{a}}_x E_{\mathrm{R}} e^{j(\omega t - \beta z)} + \hat{\mathbf{a}}_y E_{\mathrm{R}} e^{j(\omega t - \beta z - \pi/2)}\right] \\ &= E_{\mathrm{R}} \mathrm{Re}\left\{\left[\hat{\mathbf{a}}_x - j\,\hat{\mathbf{a}}_y\right] e^{j(\omega t - \beta z)}\right\}\end{aligned} \tag{4-52d}$$

We note that there is a 90° phase difference between the two orthogonal components of the electric field vector.

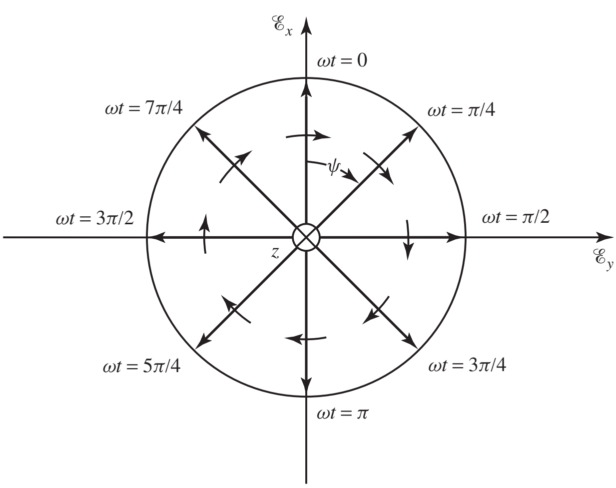


Figure 4-13 Right-hand circularly polarized wave.

Example 4-7

If $\phi_x = +\pi/2$, $\phi_y = 0$, and $E_{x0}^{+} = E_{y0}^{+} = E_R$, determine the polarization and sense of rotation of the wave of (4-50a).

Solution: Since

$$\begin{aligned}\phi_x &= +\frac{\pi}{2}\\ \phi_y &= 0\\ E_{x0}^{+} &= E_{y0}^{+} = E_R\end{aligned}$$

then

$$\begin{aligned}\mathscr{E}_x &= E_R \cos\left(\omega t + \frac{\pi}{2}\right) = -E_R \sin \omega t\\ \mathscr{E}_y &= E_R \cos(\omega t)\end{aligned}$$

and the locus of the amplitude of the electric field vector is given by

$$\mathscr{E} = \sqrt{\mathscr{E}_x^2 + \mathscr{E}_y^2} = \sqrt{E_R^2(\cos^2 \omega t + \sin^2 \omega t)} = E_R$$

The angle ψ along which the field is directed is given by

$$\psi = \tan^{-1}\left[\frac{\mathscr{E}_y}{\mathscr{E}_x}\right] = \tan^{-1}\left[-\frac{E_R \cos(\omega t)}{E_R \sin(\omega t)}\right] = \tan^{-1}[-\cot(\omega t)] = \omega t + \frac{\pi}{2}$$

The locus of the field vector is a circle of radius E_R, and it rotates clockwise with an angular frequency ω as shown in Figure 4-14; *hence, it is a right-hand circular polarization*.

The expression for the instantaneous electric field vector is

$$\begin{aligned}\boldsymbol{\mathscr{E}} &= \text{Re}\left[\hat{\mathbf{a}}_x E_R e^{j(\omega t - \beta z + \pi/2)} + \hat{\mathbf{a}}_y E_R e^{j(\omega t - \beta z)}\right]\\ &= E_R \text{Re}\left\{\left[j\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y\right] e^{j(\omega t - \beta z)}\right\}\end{aligned}$$

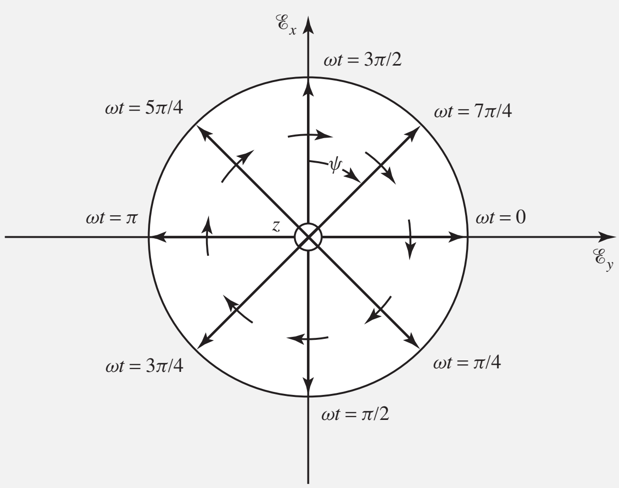


Figure 4-14 Right-hand circularly polarized wave.

Again we note a 90° phase difference between the orthogonal components.

From the previous discussion we see that a right-hand *circular polarization* can be achieved *if and only if its two orthogonal linearly polarized components have equal amplitudes and a* 90° *phase difference of one relative to the other. The sense of rotation* (*clockwise here*) *is determined by rotating the phase-leading component* (*in this instance* $\mathscr{E}_x$) *toward the phase-lagging component* (*in this instance* $\mathscr{E}_y$). *The field rotation must be viewed as the wave travels away from the observer*.

B. Left-Hand (Counterclockwise) Circular Polarization If the electric field vector has a counterclockwise sense of rotation, the polarization is designated as *left-hand polarization*. To demonstrate this, let in (4-50a)

$$\begin{aligned}\phi_x &= 0\\ \phi_y &= \frac{\pi}{2}\\ E_{x_0}^+ &= E_{y_0}^+ = E_\mathrm{L}\end{aligned} \tag{4-53}$$

then

$$\begin{aligned}\mathscr{E}_x &= E_\mathrm{L}\cos(\omega t)\\ \mathscr{E}_y &= E_\mathrm{L}\cos\left(\omega t + \frac{\pi}{2}\right) = -E_\mathrm{L}\sin(\omega t)\end{aligned} \tag{4-53a}$$

and the locus of the amplitude is

$$\mathscr{E} = \sqrt{\mathscr{E}_x^2 + \mathscr{E}_y^2} = \sqrt{E_\mathrm{L}^2(\cos^2\omega t + \sin^2\omega t)} = E_\mathrm{L} \tag{4-53b}$$

The angle ψ is given by

$$\psi = \tan^{-1}\left[\frac{\mathscr{E}_y}{\mathscr{E}_x}\right] = \tan^{-1}\left[\frac{-E_\mathrm{L}\sin(\omega t)}{E_\mathrm{L}\cos(\omega t)}\right] = -\omega t \tag{4-53c}$$

The locus of the field vector is a circle of radius E_L, and it rotates counterclockwise with an angular frequency ω as shown in Figure 4-15; *hence, it is a left-hand circular polarization*.

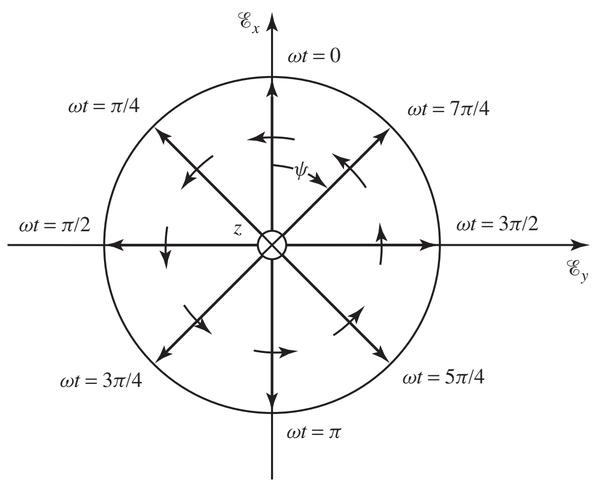


Figure 4-15 Left-hand circularly polarized wave.

The instantaneous electric field vector can be written as

$$\begin{aligned}\boldsymbol{\mathscr{E}} &= \text{Re}\left[\hat{\mathbf{a}}_x E_{\text{L}} e^{j(\omega t-\beta z)} + \hat{\mathbf{a}}_y E_{\text{L}} e^{j(\omega t-\beta z+\pi/2)}\right] \\ &= E_{\text{L}}\text{Re}\left\{\left[\hat{\mathbf{a}}_x + j\hat{\mathbf{a}}_y\right] e^{j(\omega t-\beta z)}\right\} \end{aligned} \tag{4-53d}$$

In (4-53d) we note a 90° phase advance of the $\mathscr{E}_y$ component relative to the $\mathscr{E}_x$ component.

Example 4-8

Determine the polarization and sense of rotation of the wave given by (4-50a) if $\phi_x = -\pi/2$, $\phi_y = 0$, and $E_{x0}^{+} = E_{y0}^{+} = E_{\text{L}}$.

Solution: Since

$$\begin{aligned}\phi_x &= -\frac{\pi}{2} \\ \phi_y &= 0 \\ E_{x0}^{+} &= E_{y0}^{+} = E_{\text{L}}\end{aligned}$$

then

$$\begin{aligned}\mathscr{E}_x &= E_{\text{L}}\cos\left(\omega t - \frac{\pi}{2}\right) = E_{\text{L}}\sin(\omega t) \\ \mathscr{E}_y &= E_{\text{L}}\cos(\omega t)\end{aligned}$$

and the locus of the amplitude is

$$\mathscr{E} = \sqrt{\mathscr{E}_x^2 + \mathscr{E}_y^2} = \sqrt{E_{\text{L}}^2(\sin^2\omega t + \cos^2\omega t)} = E_{\text{L}}$$

The angle ψ is given by

$$\psi = \tan^{-1}\left[\frac{\mathscr{E}_y}{\mathscr{E}_x}\right] = \tan^{-1}\left[\frac{E_{\text{L}}\cos(\omega t)}{E_{\text{L}}\sin(\omega t)}\right] = \tan^{-1}[\cot(\omega t)] = \frac{\pi}{2} - \omega t$$

The locus of the electric field vector is a circle of radius E_{L}, and it rotates counterclockwise with an angular frequency ω as shown in Figure 4-16; *hence, it is a left-hand circular polarization*. For this case we can write the electric field as

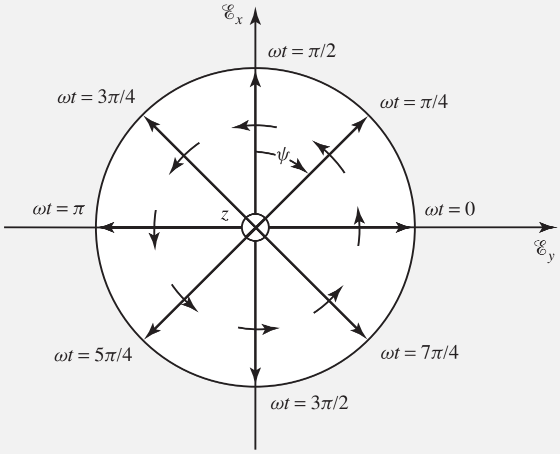


Figure 4-16 Left-hand circularly polarized wave.

$$\boldsymbol{\mathscr{E}} = \operatorname{Re}\left[\hat{\mathbf{a}}_x E_{\mathrm{L}} e^{j(\omega t - \beta z - \pi/2)} + \hat{\mathbf{a}}_y E_{\mathrm{L}} e^{j(\omega t - \beta z)}\right]$$
$$= E_{\mathrm{L}} \operatorname{Re}\left\{\left[-j\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y\right] e^{j(\omega t - \beta z)}\right\}$$

and we note a 90° phase delay of the $\mathscr{E}_x$ component relative to $\mathscr{E}_y$.

From the previous discussion we see that *left-hand circular polarization* can be achieved *if and only if its two orthogonal components have equal amplitudes and odd multiples of* 90° *phase difference of one component relative to the other. The sense of rotation* (*counterclockwise here*) *is determined by rotating the phase-leading component* (*in this instance* $\mathscr{E}_y$) *toward the phase-lagging component* (*in this instance* $\mathscr{E}_x$). *The field rotation must be viewed as the wave travels away from the observer*.

The *necessary and sufficient* conditions for circular polarization are the following:

1. The field must have two orthogonal linearly polarized components.
2. The two components must have the same magnitude.
3. The two components must have a time-phase difference of odd multiples of 90°.

The sense of rotation is always determined by rotating the phase-leading component toward the phase-lagging component and observing the field rotation as the wave is traveling away from the observer. The rotation of the phase-leading component toward the phase-lagging component should be done along the angular separation between the two components that is less than 180°. *Phases equal to or greater than* 0° *and less than* 180° *should be considered leading whereas those equal to or greater than* 180° *and less than* 360° *should be considered lagging*.

4.4.3 Elliptical Polarization

A wave is said to be elliptically polarized if the tip of the electric field vector traces, as a function of time, an elliptical locus in space. At various instants of time the electric field vector changes continuously with time in such a manner as to describe an elliptical locus. It is right-hand elliptically polarized if the electric field vector of the ellipse rotates clockwise, and it is left-hand elliptically polarized if the electric field vector of the ellipse rotates counterclockwise [8, 10–14].

Let us examine the locus of the instantaneous electric field vector ($\boldsymbol{\mathscr{E}}$) at the $z = 0$ plane at all times. For this particular example, let in (4-50a)

$$\begin{aligned}
\phi_x &= \frac{\pi}{2} \\
\phi_y &= 0 \\
E_{x0}^{+} &= (E_{\mathrm{R}} + E_{\mathrm{L}}) \\
E_{y0}^{+} &= (E_{\mathrm{R}} - E_{\mathrm{L}})
\end{aligned} \tag{4-54}$$

Then,

$$\begin{aligned}
\mathscr{E}_x &= (E_{\mathrm{R}} + E_{\mathrm{L}}) \cos\left(\omega t + \frac{\pi}{2}\right) = -(E_{\mathrm{R}} + E_{\mathrm{L}}) \sin \omega t \\
\mathscr{E}_y &= (E_{\mathrm{R}} - E_{\mathrm{L}}) \cos(\omega t)
\end{aligned} \tag{4-54a}$$

We can write the locus for the amplitude of the electric field vector as

$$\begin{aligned}
\mathscr{E}^2 &= \mathscr{E}_x^2 + \mathscr{E}_y^2 = (E_\mathrm{R} + E_\mathrm{L})^2 \sin^2 \omega t + (E_\mathrm{R} - E_\mathrm{L})^2 \cos^2 \omega t \\
&= E_\mathrm{R}^2 \sin^2 \omega t + E_\mathrm{L}^2 \sin^2 \omega t + 2E_\mathrm{R}E_\mathrm{L} \sin^2 \omega t \\
&\quad + E_\mathrm{R}^2 \cos^2 \omega t + E_\mathrm{L}^2 \cos^2 \omega t - 2E_\mathrm{R}E_\mathrm{L} \cos^2 \omega t \\
\mathscr{E}_x^2 + \mathscr{E}_y^2 &= E_\mathrm{R}^2 + E_\mathrm{L}^2 + 2E_\mathrm{R}E_\mathrm{L}\left[\sin^2 \omega t - \cos^2 \omega t\right]
\end{aligned} \tag{4-54b}$$

However,

$$\begin{aligned}
\sin \omega t &= -\mathscr{E}_x/(E_\mathrm{R} + E_\mathrm{L}) \\
\cos \omega t &= \mathscr{E}_y/(E_\mathrm{R} - E_\mathrm{L})
\end{aligned} \tag{4-54c}$$

Substituting (4-54c) into (4-54b) reduces to

$$\left\{\frac{\mathscr{E}_x}{E_\mathrm{R} + E_\mathrm{L}}\right\}^2 + \left\{\frac{\mathscr{E}_y}{E_\mathrm{R} - \mathrm{E}_\mathrm{L}}\right\}^2 = 1 \tag{4-54d}$$

which is the equation for an ellipse with the major axis $|\mathscr{E}|_{\max} = |E_\mathrm{R} + E_\mathrm{L}|$ and the minor axis $|\mathscr{E}|_{\min} = |E_\mathrm{R} - E_\mathrm{L}|$. As time elapses, the electric vector rotates and its length varies with its tip tracing an ellipse, as shown in Figure 4-17. The maximum and minimum lengths of the electric vector are the major and minor axes, given by

$$|\boldsymbol{\mathscr{E}}|_{\max} = |E_\mathrm{R} + E_\mathrm{L}|, \quad \text{when } \omega t = (2n+1)\frac{\pi}{2}, n = 0, 1, 2, \ldots \tag{4-54e}$$

$$|\boldsymbol{\mathscr{E}}|_{\min} = |E_\mathrm{R} - E_\mathrm{L}|, \quad \text{when } \omega t = n\pi, n = 0, 1, 2, \ldots \tag{4-54f}$$

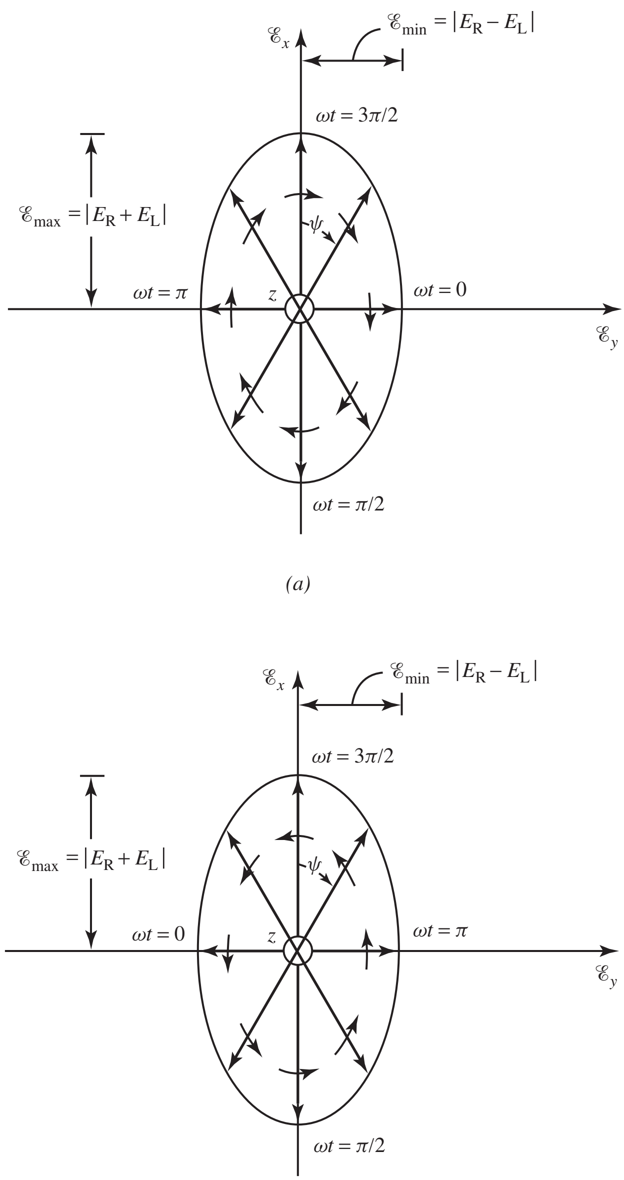
The axial ratio (AR) is defined to be the ratio of the major axis (including its sign) of the polarization ellipse to the minor axis, or

$$\mathrm{AR} = -\frac{\mathscr{E}_{\max}}{\mathscr{E}_{\min}} = -\frac{2(E_\mathrm{R} + E_\mathrm{L})}{2(E_\mathrm{R} - E_\mathrm{L})} = -\frac{(E_\mathrm{R} + E_\mathrm{L})}{(E_\mathrm{R} - E_\mathrm{L})} \tag{4-54g}$$

where E_R and E_L are positive real quantities. As defined in (4-54g), the axial ratio AR can take positive (for left-hand polarization) or negative (for right-hand polarization) values in the range $1 \leq |\mathrm{AR}| \leq \infty$. The instantaneous electric field vector can be written as

$$\begin{aligned}
\boldsymbol{\mathscr{E}} &= \mathrm{Re}\left\{\hat{\mathbf{a}}_x \left[E_\mathrm{R} + E_\mathrm{L}\right] e^{j(\omega t - \beta z + \pi/2)} + \hat{\mathbf{a}}_y \left[E_\mathrm{R} - E_\mathrm{L}\right] e^{j(\omega t - \beta z)}\right\} \\
&= \mathrm{Re}\left\{\left[\hat{\mathbf{a}}_x j(E_\mathrm{R} + E_\mathrm{L}) + \hat{\mathbf{a}}_y (E_\mathrm{R} - E_\mathrm{L})\right] e^{j(\omega t - \beta z)}\right\} \\
\boldsymbol{\mathscr{E}} &= \mathrm{Re}\left\{\left[E_\mathrm{R}(j\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y) + E_\mathrm{L}(j\hat{\mathbf{a}}_x - \hat{\mathbf{a}}_y)\right] e^{j(\omega t - \beta z)}\right\}
\end{aligned} \tag{4-54h}$$

From (4-54h) we see that we can represent an elliptical wave as the sum of a right-hand [first term of (4-54h)] and a left-hand [second term of (4-54h)] circularly polarized waves with amplitudes E_R and E_L, respectively. If $E_\mathrm{R} > \mathrm{E}_L$, the axial ratio will be negative and the right-hand circular component will be stronger than the left-hand circular component. Thus, the electric vector rotate in the same direction as that of the right-hand circularly polarized wave, producing a *right-hand elliptically polarized wave*, as shown in Figure 4-17*a*. If $E_\mathrm{L} > E_\mathrm{R}$, the axial ratio will be positive and the left-hand circularly polarized component will be stronger than the right-hand circularly polarized component. The electric field vector will rotate in the same direction as that of the left-hand circularly polarized component, producing a *left-hand elliptically polarized wave*, as shown in Figure 4-17*b*. The sign of the axial ratio carries information on the direction of rotation of the electric field vector.



(b)

Figure 4-17 Right- and left-hand elliptical polarizations with major axis along the x axis. (a) Right-hand (clockwise) when $E_R > E_L$. (b) Left-hand (counterclockwise) when $E_R < E_L$.

An analogous situation exists when

$$\begin{aligned}\phi_x &= \frac{\pi}{2}\\ \phi_y &= 0\\ E_{x_0}^{+} &= (E_R - E_L)\\ E_{y_0}^{+} &= (E_R + E_L)\end{aligned} \tag{4-55}$$

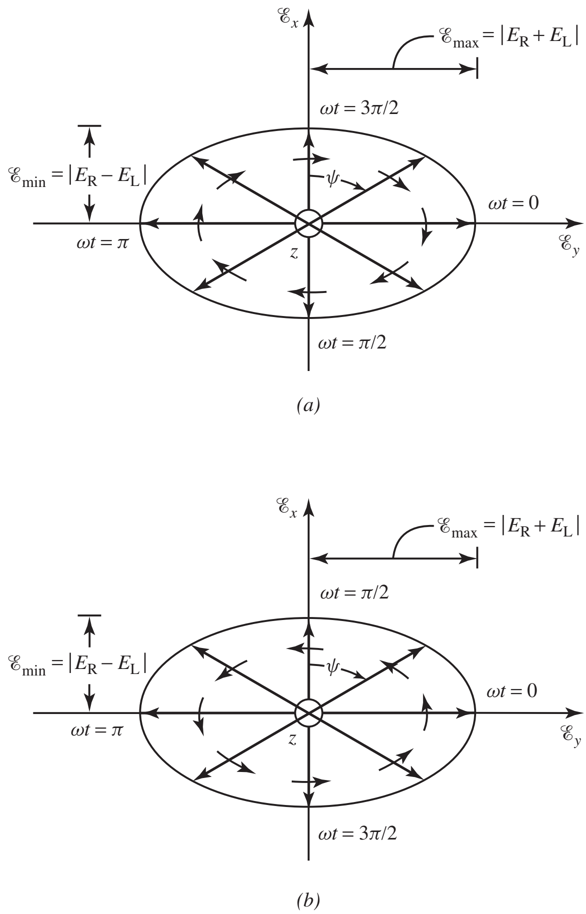


Figure 4-18 Right- and left-hand elliptical polarizations with major axis along the y axis. (*a*) Right-hand (clockwise) when $E_R > E_L$. (*b*) Left-hand (counterclockwise) when $E_R < E_L$.

The polarization loci are shown in Figure 4-18*a* and 4-18*b* when $E_R > E_L$ and $E_R < E_L$, respectively.

From (4-54e) and (4-54f), it can be seen that the component of $\boldsymbol{\mathscr{E}}$ measured along the major axis of the polarization ellipse is 90° out of phase with the component of $\boldsymbol{\mathscr{E}}$ measured along the minor axis. Also with the aid of (4-54b), it can be shown that the electric vector rotates through 90° in space between the instants of time given by (4-54e) and (4-54f) when the vector has maximum and minimum lengths, respectively. Thus the major and minor axes of the polarization ellipse are orthogonal in space, just as we might anticipate.

Since linear polarization is a special kind of elliptical polarization, we can represent a linear polarization as the sum of a right- and a left-hand circularly polarized components of *equal amplitudes*. We see that for this case ($E_R = E_L$), (4-54h) will degenerate into a linear polarization.

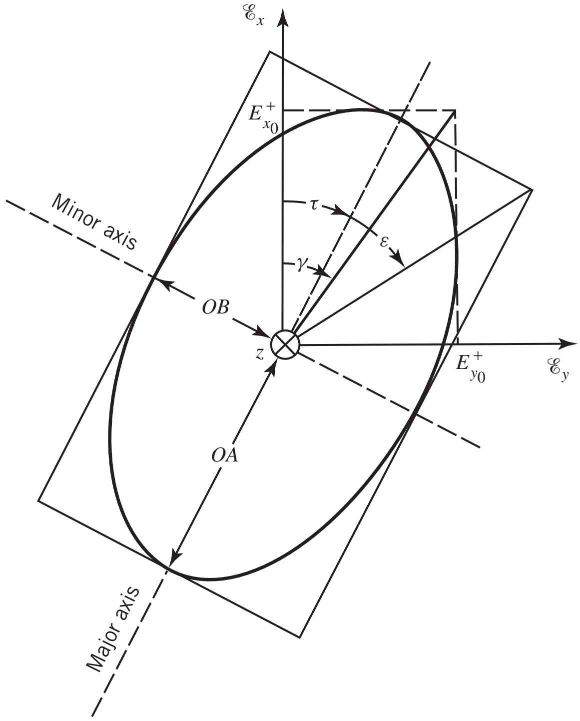


Figure 4-19 Rotation of a plane electromagnetic wave and its tilted ellipse at $z = 0$ as a function of time.

A more general orientation of an elliptically polarized locus is the tilted ellipse of Figure 4-19. This is representative of the fields of (4-50a) when

$$\Delta\phi = \phi_x - \phi_y \neq \frac{n\pi}{2} \qquad n = 0, 2, 4 \ldots$$

$$\geq 0 \quad \begin{cases} \text{for CW if } E_R > E_L \\ \text{for CCW if } E_R < E_L \end{cases} \tag{4-56a}$$

$$\leq 0 \quad \begin{cases} \text{for CW if } E_R < E_L \\ \text{for CCW if } E_R > E_L \end{cases} \tag{4-56b}$$

$$E_{x_0}^{+} = E_R + E_L$$

$$E_{y_0}^{+} = E_R - E_L \tag{4-56c}$$

Thus the major and minor axes of the ellipse do not, in general, coincide with the principal axes of the coordinate system unless the magnitudes are not equal and the phase difference between the two orthogonal components is equal to odd multiples of $\pm 90^\circ$.

The ratio of the major to the minor axes, which is defined as the axial ratio (AR), is equal to [8]

$$\text{AR} = \pm\frac{\text{major axis}}{\text{minor axis}} = \pm\frac{OA}{OB}, \qquad 1 \le |\text{AR}| \le \infty \tag{4-57}$$

where

$$OA = \left[\frac{1}{2}\left\{(E_{x0}^{+})^2 + (E_{y0}^{+})^2 + \left[(E_{x0}^{+})^4 + (E_{y0}^{+})^4 + 2(E_{x0}^{+})^2(E_{y0}^{+})^2\cos(2\Delta\phi)\right]^{1/2}\right\}\right]^{1/2} \tag{4-57a}$$

$$OB = \left[\frac{1}{2}\left\{(E_{x0}^{+})^2 + (E_{y0}^{+})^2 - \left[(E_{x0}^{+})^4 + (E_{y0}^{+})^4 + 2(E_{x0}^{+})^2(E_{y0}^{+})^2\cos(2\Delta\phi)\right]^{1/2}\right\}\right]^{1/2} \tag{4-57b}$$

E_{x0}^{+} and E_{y0}^{+} are given by (4-56c). The plus (+) sign in (4-57) is for left-hand and the minus (−) sign is for right-hand polarization.

The tilt of the ellipse, *relative to the x axis*, is represented by the angle τ given by

$$\tau = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left[\frac{2E_{x0}^{+}E_{y0}^{+}}{(E_{x0}^{+})^2 - (E_{y0}^{+})^2}\cos(\Delta\phi)\right] \tag{4-57c}$$

4.4.4 Poincaré Sphere

The polarization state, defined here as P, of any wave can be uniquely represented by a point on the surface of a sphere [15–19]. This is accomplished by either of the two pairs of angles (γ, δ) or (ε, τ). By referring to (4-50a) and Figure 4-20a, we can define the two pairs of angles:

$$\underline{(\gamma, \delta)\ \text{set}}$$

$$\gamma = \tan^{-1}\left[\frac{E_{y0}^{+}}{E_{x0}^{+}}\right] \quad \text{or} \quad \gamma = \tan^{-1}\left[\frac{E_{x0}^{+}}{E_{y0}^{+}}\right], \qquad 0^\circ \le \gamma \le 90^\circ \tag{4-58a}$$

$$\delta = \phi_y - \phi_x = \text{phase difference between } \mathscr{E}_y \text{ and } \mathscr{E}_x, \quad -180^\circ \le \delta \le 180^\circ \tag{4-58b}$$

where 2γ is the great-circle angle drawn from a reference point on the equator and δ is the equator to great-circle angle;

$$\underline{(\varepsilon, \tau)\ \text{set}}$$

$$\varepsilon = \cot^{-1}(\text{AR}) \Rightarrow \text{AR} = \cot(\varepsilon), \quad -45^\circ \le \varepsilon \le +45^\circ \tag{4-59a}$$

$$\tau = \text{tilt angle}, \qquad 0^\circ \le \tau \le 180^\circ \tag{4-59b}$$

where

$$2\varepsilon = \text{latitude}$$

$$2\tau = \text{longitude}$$

In (4-58a) *the appropriate ratio is the one that satisfies the angular limits of all the Poincaré sphere angles* (*especially those of* ε). *The axial ratio AR is positive for left-hand polarization and*

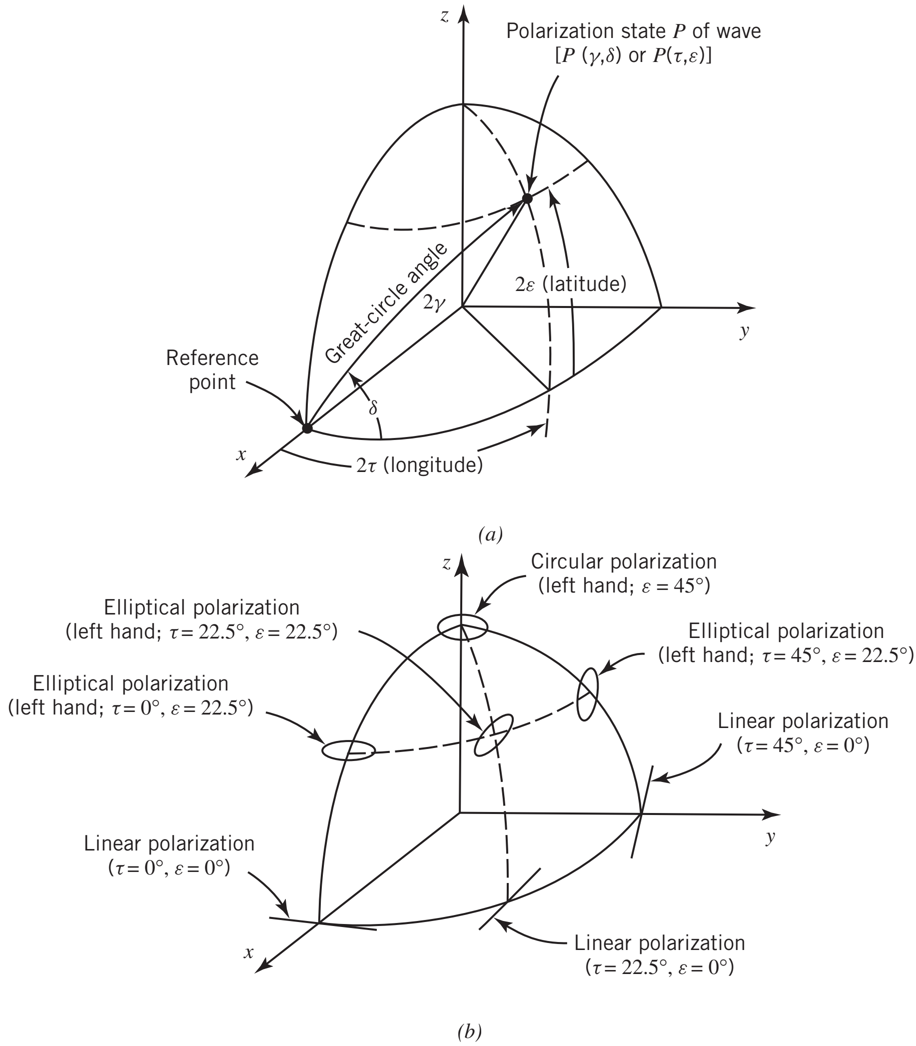


Figure 4-20 Poincaré sphere for the polarization state of an electromagnetic wave. (Source: J. D. Kraus, *Electromagnetics*, 1984, McGraw-Hill Book Co.). (*a*) Poincaré sphere. (*b*) Polarization state.

negative for right-hand polarization. Some polarization states are displayed on the first octant of the Poincaré sphere in Figure 4-20*b*. The polarization states on a planar surface representation (projection) of the Poincaré sphere ($-45° \leq \varepsilon \leq +45°$, $0° \leq \tau \leq 180°$) are shown in Figure 4-21.

For the polarization ellipse of Figure 4-19, the two sets of angles are related geometrically as shown in Figure 4-20. Analytically, it can be shown through spherical trigonometry [20] that the two pairs of angles (γ, δ) and (ε, τ) are related by

$$\boxed{\begin{aligned} \cos(2\gamma) &= \cos(2\varepsilon)\cos(2\tau) \\ \tan(\delta) &= \frac{\tan(2\varepsilon)}{\sin(2\tau)} \end{aligned}} \qquad \begin{aligned} &(4\text{-}60\text{a}) \\ &(4\text{-}60\text{b}) \end{aligned}$$

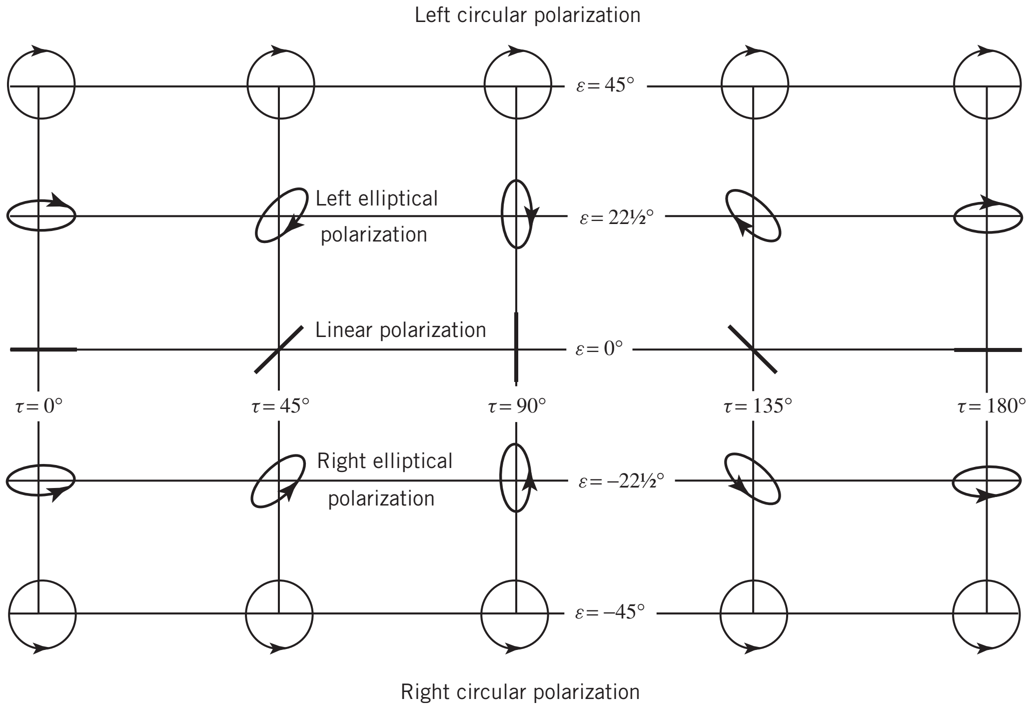


Figure 4-21 Polarization states of electromagnetic waves on a planar surface projection of a Poincaré sphere. (Source: J. D. Kraus, *Electromagnetics*, 1984, McGraw-Hill Book Co.).

or

$$\boxed{\begin{aligned} \sin(2\varepsilon) &= \sin(2\gamma)\sin(\delta) \\ \tan(2\tau) &= \tan(2\gamma)\cos(\delta) \end{aligned}} \qquad \begin{aligned} &(4\text{-}61\text{a}) \\ &(4\text{-}61\text{b}) \end{aligned}$$

Thus one set can be obtained by knowing the other.

It is apparent from Figure 4-20 that the linear polarization is always found along the equator; the right-hand circular resides along the south pole and the left-hand circular along the north pole. The remaining surface of the sphere is used to represent elliptical polarization with left-hand elliptical in the upper hemisphere and right-hand elliptical on the lower hemisphere.

Because the Poincaré sphere parameter pairs (γ, δ) and (ε, τ) are related by transcendental functions, of (4-60a) and (4-60b), there may be some ambiguity at which quadrant should the angles be chosen. The angles should be selected to each satisfy respectively the range of values given by (4-58a) and (4-58b), and (4-57c), and each set should represent the same point on the Poincaré sphere. Also the range of values of the axial ratio (AR) should be $1 \leq |\text{AR}| \leq \infty$, with positive values to represent CCW (left-hand) polarization and negative values to represent CW (right-hand) polarization. A MATLAB computer program, **Polarization_Propag**, has been written and it is part of the website that accompanies this book.

Example 4-9

Determine the point on the Poincaré sphere of Figure 4-20 when the wave represented by (4-50a) is such that

$$\begin{aligned}\mathscr{E}_x &= E_{x0}^{+}\cos(\omega t - \beta z + \phi_x)\\ \mathscr{E}_y &= 0\end{aligned}$$

Solution: Using (4-58a) and (4-58b)

$$\gamma = \tan^{-1}\left[\frac{E_{y0}^{+}}{E_{x0}^{+}}\right] = \tan^{-1}\left[\frac{0}{E_{x0}^{+}}\right] = 0^\circ$$

and δ could be of any value, i.e., $-180^\circ \leq \delta \leq 180^\circ$. The values of ε and τ can now be obtained from (4-61a) and (4-61b), and they are equal to

$$\begin{aligned}2\varepsilon &= \sin^{-1}\left[\sin(2\gamma)\sin(\delta)\right] = \sin^{-1}(0) = 0^\circ\\ 2\tau &= \tan^{-1}\left[\tan(2\gamma)\cos(\delta)\right] = \tan^{-1}(0) = 0^\circ\end{aligned}$$

It is apparent that for this wave, which is obviously linearly polarized, the polarization state (point) is at the reference point of Figure 4-20. The axial ratio is obtained from (4-59a), and it is equal to

$$\text{AR} = \cot(\varepsilon) = \cot(0) = \infty$$

An axial ratio of infinity always represents linear polarization.

Example 4-10

Repeat Example 4-9 when the wave of (4-50a) is such that

$$\begin{aligned}\mathscr{E}_x &= 0\\ \mathscr{E}_y &= E_{y0}^{+}\cos(\omega t - \beta z + \phi_y)\end{aligned}$$

Solution: Using (4-58a) and (4-58b),

$$\gamma = \tan^{-1}\left[\frac{E_{y0}^{+}}{E_{x0}^{+}}\right] = \tan^{-1}(\infty) = 90^\circ$$

and δ could be of any value, i.e., $-180^\circ \leq \delta \leq 180^\circ$. The values of ε and τ can now be obtained from (4-61a) and (4-61b), and they are equal to

$$\begin{aligned}2\varepsilon &= \sin^{-1}\left[\sin(2\gamma)\sin(\delta)\right] = \sin^{-1}(0) = 0^\circ\\ 2\tau &= \tan^{-1}\left[\tan(2\gamma)\cos(\delta)\right] = \tan^{-1}(0) = 180^\circ\end{aligned}$$

The polarization state (point) of this linearly polarized wave is diametrically opposed to that in Example 4-9. The axial ratio is also infinity.

Example 4-11

Determine the polarization state (point) on the Poincaré sphere of Figure 4-20 when the wave of (4-50a) is such that

$$\mathscr{E}_x = E_{x0}^+ \cos(\omega t - \beta z + \phi_x) = 2E_0 \cos\left(\omega t - \beta z + \frac{\pi}{2}\right)$$

$$\mathscr{E}_y = E_{y0}^+ \cos(\omega t - \beta z + \phi_y) = E_0 \cos(\omega t - \beta z)$$

Solution: Using (4-58a) and (4-58b),

$$\gamma = \tan^{-1}\left[\frac{E_{y0}^+}{E_{x0}^+}\right] = \tan^{-1}\left[\frac{E_0}{2E_0}\right] = 26.56^\circ$$

$$\delta = \phi_y - \phi_x = -90^\circ$$

The values of ε and τ can now be obtained from (4-61a) and (4-61b), and they are equal to

$$2\varepsilon = \sin^{-1}\left[\sin(2\gamma)\sin(\delta)\right] = \sin^{-1}\left[-\sin(2\gamma)\right] = -2\gamma = -53.12^\circ$$

$$2\tau = \tan^{-1}\left[\tan(2\gamma)\cos(\delta)\right] = \tan^{-1}(0) = 0^\circ$$

Therefore, this point is situated on the principal xz plane at an angle of $2\gamma = -2\varepsilon = 53.12^\circ$ from the reference point of the x axis of Figure 4-20. The axial ratio is obtained using (4-59a), and it is equal to

$$\text{AR} = \cot(\varepsilon) = \cot(-26.56^\circ) = -2$$

The negative sign indicates that the wave has a right-hand (clockwise) polarization. Therefore the wave is right-hand elliptically polarized with AR $= -2$.

In general, points on the principal xz elevation plane, aside from the two intersecting points on the equator and the north and south poles, are used to represent elliptical polarization when the major and minor axes of the polarization ellipse of Figure 4-19 coincide with the principal axes.

If the polarization state of a wave is defined as P_w and that of an antenna as P_a, then the voltage response of the antenna due to the wave is obtained by [10, 19]

$$V = C \cos\left[\frac{P_w P_a}{2}\right] \tag{4-62}$$

where

$C =$ constant that is a function of the antenna size and field strength of the wave
$P_w =$ polarization state of the wave
$P_a =$ polarization state of the antenna
$P_w P_a =$ angle subtended by a great-circle arc from polarization P_w to P_a

Remember that the polarization of a wave, by IEEE standards [7, 8], is determined as the wave is observed from the rear (is receding). Therefore the polarization of the antenna is determined by its radiated field in the transmitting mode.

Example 4-12

If the polarization states of the wave and antenna are given, respectively, by those of Examples 4-9 and 4-10, determine the voltage response of the antenna due to that wave.

Solution: Since the polarization state P_w of the wave is at the $+x$ axis and that of the antenna P_a is at the $-x$ axis of Figure 4-20, then the angle $P_w P_a$ subtended by a great-circle arc from P_w to P_a is equal to

$$P_w P_a = 180^\circ$$

Therefore the voltage response of the antenna is, according to (4-62), equal to

$$V = C \cos\left[\frac{P_w P_a}{2}\right] = C \cos(90^\circ) = 0$$

This is expected since the fields of the wave and those of the antenna are orthogonal (cross-polarized) to each other.

Example 4-13

The polarization of a wave that impinges upon a left-hand (counterclockwise) circularly polarized antenna is circularly polarized. Determine the response of the antenna when the sense of rotation of the incident wave is

1. Left-hand (counterclockwise).
2. Right-hand (clockwise).

Solution:

1. Since the antenna is left-hand circularly polarized, its polarization state (point) on the Poincaré sphere is on the north pole ($2\gamma = \delta = 90^\circ$). When the wave is also left-hand circularly polarized, its polarization state (point) is also on the north pole ($2\gamma = \delta = 90^\circ$). Therefore, the subtended angle $P_w P_a$ between the two polarization states is equal to

   $$P_w P_a = 0^\circ$$

   and the voltage response of the antenna, according to (4-62), is equal to

   $$V = C \cos\left[\frac{P_w P_a}{2}\right] = C \cos(0) = C$$

   This represents the maximum response of the antenna, and it occurs when the polarization (including sense of rotation) of the wave is the same as that of the antenna.
2. When the sense of rotation of the wave is right-hand circularly polarized, its polarization state (point) is on the south pole ($2\gamma = 90^\circ$, $\delta = -90^\circ$). Therefore, the subtended angle $P_w P_a$ between the two polarization states is equal to

   $$P_w P_a = 180^\circ$$

   and the response of the antenna, according to (4-62), is equal to

$$V = C \cos\left[\frac{P_w P_a}{2}\right] = C \cos\left[\frac{180^\circ}{2}\right] = C \cos(90^\circ) = 0$$

This represents a null response of the antenna, and it occurs when the sense of rotation of the circularly polarized wave is opposite to that of the circularly polarized antenna. This is one technique, in addition to those shown in Example 4-12, that can be used to null the response of an antenna system.

4.5 MULTIMEDIA

On the website that accompanies this book, the following multimedia resources are included for the review, understanding and presentation of the material of this chapter.

- **MATLAB** computer programs:
  a. **Polarization_Diagram_Ellipse_Animation:** Animates the 3-D polarization diagram of a rotating electric field vector (Figure 4-8). It also animates the 2-D polarization ellipse (Figure 4-19) for linear, circular and elliptical polarized waves, and sense of rotation. It also computes the axial ratio (AR).
  b. **Polarization_Propag:** Computes the Poincaré sphere angles, and thus the polarization wave traveling in an infinite homogeneous medium.
- **Power Point (PPT)** viewgraphs, in multicolor.

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PROBLEMS

4.1. A uniform plane wave having only an x component of the electric field is traveling in the $+z$ direction in an unbounded lossless, source-free region. Using Maxwell's equations write expressions for the electric and corresponding magnetic field intensities. Compare your answers to those of (4-2b) and (4-3c).

4.2. Using Maxwell's equations, find the magnetic field components for the wave whose electric field is given in Example 4-1. Compare your answer with that obtained in the solution of Example 4-1.

4.3. The complex **H** field of a uniform plane wave, traveling in an unbounded source-free medium of free space, is given by

$$\mathbf{H} = \frac{1}{120\pi}(\hat{\mathbf{a}}_x - 2\hat{\mathbf{a}}_y)e^{-j\beta_0 z}$$

Find the:
(a) Corresponding electric field.
(b) Instantaneous power density vector.
(c) Time-average power density.

4.4. The complex **E** field of a uniform plane wave is given by

$$\mathbf{E} = (\hat{\mathbf{a}}_x + j\hat{\mathbf{a}}_z)e^{-j\beta_0 y} + (2\hat{\mathbf{a}}_x - j\hat{\mathbf{a}}_z)e^{+j\beta_0 y}$$

Assuming an unbounded source-free, free-space medium, find the:
(a) Corresponding magnetic field.
(b) Time-average power density flowing in the $+y$ direction.
(c) Time-average power density flowing in the $-y$ direction.

4.5. The magnetic field of a uniform plane wave in a source-free region is given by

$$\mathbf{H} = 10^{-6}\left[-\hat{\mathbf{a}}_x(2+j) + \hat{\mathbf{a}}_z(1+j3)\right]e^{+j\beta y}$$

Assuming that the medium is free space, determine the:
(a) Corresponding electric field.
(b) Time-average power density.

4.6. The electric field of a uniform plane wave traveling in a source-free region of free space is given by

$$\mathbf{E} = 10^{-3}(\hat{\mathbf{a}}_x + j\hat{\mathbf{a}}_y)\sin(\beta_0 z)$$

(a) Is this a traveling or a standing wave?
(b) Identify the traveling wave(s) of the electric field and the direction(s) of travel.
(c) Find the corresponding magnetic field.
(d) Determine the time-average power density of the wave.

4.7. The magnetic field of a uniform plane wave traveling in a source-free, free-space region is given by

$$\mathbf{H} = 10^{-6}(\hat{\mathbf{a}}_y + j\hat{\mathbf{a}}_z)\cos(\beta_0 x)$$

(a) Is this a traveling or a standing wave?
(b) Identify the traveling wave(s) of the magnetic field and the direction(s) of travel.
(c) Find the corresponding electric field.
(d) Determine the time-average power density of the wave.

4.8. A uniform plane wave is traveling in the $-z$ direction inside an unbounded source-free, free-space region. Assuming that the electric field has only an E_x component, its value at $z = 0$ is 4×10^{-3} V/m, and its frequency of operation is 300 MHz, write expressions for the:
(a) Complex electric and magnetic fields.
(b) Instantaneous electric and magnetic fields.
(c) Time-average and instantaneous power densities.
(d) Time-average and instantaneous electric and magnetic energy densities.

4.9. A uniform plane wave traveling inside an unbounded free-space medium has peak electric and magnetic fields given by

$$\mathbf{E} = \hat{\mathbf{a}}_x E_0 e^{-j\beta_0 z}$$
$$\mathbf{H} = \hat{\mathbf{a}}_y H_0 e^{-j\beta_0 z}$$

where $E_0 = 1$ mV/m.
(a) Evaluate H_0.
(b) Find the corresponding average power density. Evaluate all the constants.
(c) Determine the volume electric and magnetic energy densities. Evaluate all the constants.

4.10. The complex electric field of a uniform plane wave traveling in an unbounded nonferromagnetic dielectric medium is given by

$$\mathbf{E} = \hat{\mathbf{a}}_y 10^{-3} e^{-j2\pi z}$$

where z is measured in meters. Assuming that the frequency of operation is 100 MHz, find the:
(a) Phase velocity of the wave (give units).
(b) Dielectric constant of the medium.
(c) Wavelength (in meters).
(d) Time-average power density.
(e) Time-average total energy density.

4.11. The complex electric field of a time-harmonic field in free space is given by

$$\mathbf{E} = \hat{\mathbf{a}}_z 10^{-3}(1 + j)e^{-j(2/3)\pi x}$$

Assuming the distance x is measured in meters, find the:
(a) Wavelength (in meters).
(b) Frequency.
(c) Associated magnetic field.

4.12. A uniform plane wave is traveling inside the earth, which is assumed to be a perfect dielectric infinite in extent. If the relative permittivity of the earth is 9, find, at a frequency of 1 MHz, the:
(a) Phase velocity.
(b) Wave impedance.
(c) Intrinsic impedance.
(d) Wavelength of the wave inside the earth.

4.13. An 11-GHz transmitter radiates its power isotropically in a free-space medium. Assuming its total radiated power is 50 mW, at a distance of 3 km, find the:
(a) Time-average power density.
(b) RMS electric and magnetic fields.
(c) Total time-average volume energy densities.
In all cases, specify the units.

4.14. The electric field of a time-harmonic wave traveling in free space is given by

$$\mathbf{E} = \hat{\mathbf{a}}_x 10^{-4}(1 + j)e^{-j\beta_0 z}$$

Find the amount of real power crossing a rectangular aperture whose cross section is perpendicular to the z axis. The area of the aperture is 20 cm^2.

4.15. The following complex electric field of a time-harmonic wave traveling in a source-free, free-space region is given by

$$\mathbf{E} = 5 \times 10^{-3}(4\hat{\mathbf{a}}_y + 3\hat{\mathbf{a}}_z)e^{j(6y-8z)}$$

Assuming y and z represent their respective distances in meters, determine the:
(a) Angle of wave travel (relative to the z axis).
(b) Three phase constants of the wave along its oblique direction of travel, the y axis, and the z axis (in radians per meter).
(c) Three wavelengths of the wave along its oblique direction of travel, the y axis, and the z axis (in meters).
(d) Three phase velocities of the wave along the oblique direction of travel, the y axis, and the z axis (in meters per second).
(e) Three energy velocities of the wave along the oblique direction of travel, the y axis, and the z axis (in meters per second).
(f) Frequency of the wave.
(g) Associated magnetic field.

4.16. Using Maxwell's equations, determine the magnetic field of (4-18b) given the electric field of (4-18a).

4.17. Given the electric field of Example 4-2 and using Maxwell's equations, determine the magnetic field. Compare it with that found in the solution of Example 4-2.

4.18. Given (4-19a) and (4-19c), determine the phase velocities of (4-22) and (4-23).

4.19. Derive the energy velocity of (4-24) using the definition of (4-9), (4-18a), and (4-18b).

4.20. A uniform plane wave of 3 GHz is incident upon an unbounded conducting medium of copper that has a conductivity of 5.76×10^7 S/m, $\varepsilon = \varepsilon_0$, and $\mu = \mu_0$. Find the approximate:
(a) Intrinsic impedance of copper.
(b) Skin depth (in meters).

4.21. The magnetic field intensity of a plane wave traveling in a lossy earth is given by

$$\mathbf{H} = (\hat{\mathbf{a}}_y + j2\hat{\mathbf{a}}_z)H_0 e^{-\alpha x} e^{-j\beta x}$$

where $H_0 = 1\,\mu$A/m. Assuming the lossy earth has a conductivity of 10^{-4} S/m, a dielectric constant of 9, and the frequency of operation is 1 GHz, find inside the earth the:
(a) Corresponding electric field vector.
(b) Average power density vector.
(c) Phase constant (radians per meter).
(d) Phase velocity (meters per second).
(e) Wavelength (meters).
(f) Attenuation constant (Nepers per meter).
(g) Skin depth (meters).

4.22. Sea water is an important medium in communication between submerged submarines or between submerged submarines and receiving and transmitting stations located above the surface of the sea. Assuming the constitutive electrical parameters of the sea are $\sigma = 4$ S/m, $\varepsilon_r = 81$, $\mu_r = 1$, and $f = 10^4$ Hz, find the:
(a) Complex propagation constant (per meter).
(b) Phase velocity (meters per second).
(c) Wavelength (meters).
(d) Attenuation constant (Nepers per meter).
(e) Skin depth (meters).

4.23. The electrical constitutive parameters of moist earth at a frequency of 1 MHz are $\sigma = 10^{-1}$ S/m, $\varepsilon_r = 4$, and $\mu_r = 1$. Assuming that the electric field of a uniform plane wave at the interface (on the side of the earth) is 3×10^{-2} V/m, find the:
(a) Distance through which the wave must travel before the magnitude of the electric field reduces to 1.104×10^{-2} V/m.
(b) Attenuation the electric field undergoes in part (a) (in decibels).
(c) Wavelength inside the earth (in meters).
(d) Phase velocity inside the earth (in meters per second).
(e) Intrinsic impedance of the earth.

4.24. The complex electric field of a uniform plane wave is given by

$$\mathbf{E} = 10^{-2}\left[\hat{\mathbf{a}}_x\sqrt{2} + \hat{\mathbf{a}}_z(1+j)e^{j\pi/4}\right]e^{-j\beta y}$$

(a) Find the polarization of the wave (linear, circular, or elliptical).
(b) Determine the sense of rotation (clockwise or counterclockwise).
(c) Sketch the figure the electric field traces as a function of ωt.

4.25. The complex magnetic field of a uniform plane wave is given by

$$\mathbf{H} = \frac{10^{-3}}{120\pi}(\hat{\mathbf{a}}_x - j\hat{\mathbf{a}}_z)e^{+j\beta y}$$

(a) Find the polarization of the wave (linear, circular, or elliptical).
(b) State the direction of rotation (clockwise or counterclockwise). Justify your answer.
(c) Sketch the polarization curve denoting the $\mathcal{H}$-field amplitude, and direction of rotation. Indicate on the curve the various times for the rotation of the vector.

4.26. In a source-free, free-space region, the complex magnetic field of a time-harmonic field is represented by

$$\mathbf{H} = \left[\hat{\mathbf{a}}_x(1+j) + \hat{\mathbf{a}}_z\sqrt{2}e^{j\pi/4}\right]\frac{E_0}{\eta_0}e^{-j\beta_0 y}$$

where E_0 is a constant and η_0 is the intrinsic impedance of free space. Determine the:
(a) Polarization of the wave (linear, circular, or elliptical). Justify your answer.
(b) Sense of rotation, if any.
(c) Corresponding electric field.

4.27. Show that any linearly polarized wave can be decomposed into two circularly polarized waves (one CW and the other CCW) but both traveling in the same direction as the linearly polarized wave.

4.28. The electric field of a $f = 10$ GHz time-harmonic uniform plane wave traveling in a perfect dielectric medium is given by

$$\mathbf{E} = \left(\hat{\mathbf{a}}_x + j2\hat{\mathbf{a}}_y\right)e^{-j600\pi z}$$

where z is in meters. Determine, assuming the permeability of the medium is the same as that of free space, the:
(a) Wavelength of the wave (in meters).
(b) Velocity of the wave (in meters/sec).
(c) Dielectric constant (relative permittivity) of the medium (dimensionless).
(d) Intrinsic impedance of the medium (in ohms).
(e) Wave impedance of the medium (in ohms).
(f) Vector magnetic field of the wave.
(g) Polarization of the wave (linear, circular, elliptical; AR; and sense of rotation).

4.29. The spatial variations of the electric field of a time-harmonic wave traveling in free space are given by

$$\mathbf{E}(x) = \hat{\mathbf{a}}_y e^{-j(\beta_0 x - \frac{\pi}{4})} + \hat{\mathbf{a}}_z e^{-j(\beta_0 x - \frac{\pi}{2})}$$

Determine, using the necessary and sufficient conditions of the wave, the:
(a) Direction of wave travel ($+x$, $-x$, $+y$, $-y$, $+z$ or $-z$) based on $e^{+j\omega t}$ time.
(b) Polarization of the wave (linear, circular or elliptical). Justify your answer.
(c) Sense of rotation (CW or CCW), if any, of the wave. Justify your answer.

4.30. The spatial variations of the electric field of a time-harmonic wave traveling in free space are given by

$$\mathbf{E}(z) = \hat{\mathbf{a}}_x 2e^{-j(\beta_0 z - \frac{\pi}{4})} + \hat{\mathbf{a}}_y e^{-j(\beta_0 z - \frac{3\pi}{4})}$$

Determine the:
(a) Direction of wave travel ($+x$, $-x$, $+y$, $-y$, $+z$ or $-z$) based on $e^{+j\omega t}$ time.
(b) Two pairs of Poincaré sphere polarization parameters (γ, δ) and (ε, τ).
(c) Based on either one of the two pairs of parameters from part (b), state the:
- Polarization of the wave (linear, circular or elliptical). Justify your answer.
- Sense of rotation (CW or CCW) of the wave. Justify your answer.
- Axial Ratio. Justify your answer.

4.31. The time-harmonic electric field traveling inside an infinite lossless dielectric medium is given by

$$\mathbf{E}^i(z) = \left(j2\hat{\mathbf{a}}_x + 5\hat{\mathbf{a}}_y\right) E_0 e^{-j\beta z}$$

where β and E_o are real constants. Assuming a $e^{+j\omega t}$ time convention, determine the:
(a) Polarization of the wave (linear, circular or elliptical). You must justify your answer. Be specific.
(b) Sense of rotation (CW or CCW). You must justify your answer. Be specific.
(c) Axial Ratio (AR) based on the expression of the electric field. You must justify your answer. Be specific.
(d) Poincaré sphere angles (in degrees):
- γ and δ
- ε and τ

Make sure that the polarization point on the Poincaré sphere based on the pair of angles (γ, δ) is the same as that based on the set of angles (ε, τ).
(e) Axial Ratio (AR) based on the Poincaré sphere angles. Compare with that in part (c).

4.32. In a source-free, free-space region the complex magnetic field is given by

$$\mathbf{H} = j(\hat{\mathbf{a}}_y - j\hat{\mathbf{a}}_z)\frac{E_0}{\eta_0}e^{+j\beta_0 x}$$

where E_0 is a constant and η_0 is the intrinsic impedance of free space. Find the:
(a) Polarization of the wave (linear, circular, or elliptical). Justify your answer.
(b) Sense of rotation, if any (CW or CCW). Justify your answer.
(c) Time-average power density.
(d) Polarization of the wave on the Poincaré sphere.

4.33. The electric field of a time-harmonic wave is given by

$$\mathbf{E} = 2 \times 10^{-3}(\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y)e^{-j2z}$$

(a) State the polarization of the wave (linear, circular, or elliptical).
(b) Find the polarization on the Poincaré sphere by identifying the angles δ, γ, τ and ε (in degrees).
(c) Locate the polarization point on the Poincaré sphere.

4.34. For a uniform plane wave represented by the electric field

$$\mathbf{E} = E_0(\hat{\mathbf{a}}_x - j2\hat{\mathbf{a}}_y)e^{-j\beta z}$$

where E_0 is constant, do the following.
(a) Determine the longitude angle 2τ, latitude angle 2ε, great-circle angle 2γ, and equator to great-circle angle δ (all

in degrees) that are used to identify and locate the polarization of the wave on the Poincaré sphere.

(b) Using the answers from part (a), state the polarization of the wave (linear, circular, or elliptical), its sense of rotation (CW or CCW), and its Axial Ratio.

(c) Find the signal loss (in decibels) when the wave is received by a right-hand circularly polarized antenna.

4.35. The electric field of (4-50a) has an Axial Ratio of infinity and a great-circle angle of $2\gamma = 109.47^\circ$.

(a) Find the relative magnitude (ratio) of $E_{y_0}^+$ to $E_{x_0}^+$. Which component is more dominant, E_x or E_y? Use the first definition of γ in (4-58a).

(b) Identify the polarization point on the Poincaré sphere (i.e., find δ, τ, and ε in degrees).

(c) State the polarization of the wave (linear, circular, or elliptical).

4.36. A uniform plane wave is traveling along the $+z$ axis and its electric field is given by

$$\mathbf{E}_w = (\hat{\mathbf{a}}_x + j\hat{\mathbf{a}}_y)e^{-j\beta z}E_0$$

This incident plane wave impinges upon an antenna whose field radiated along the z axis is given by

(a) $\mathbf{E}_{aa} = (\hat{\mathbf{a}}_x + j\hat{\mathbf{a}}_y)e^{+j\beta z}E_a$

(b) $\mathbf{E}_{ab} = (\hat{\mathbf{a}}_x - j\hat{\mathbf{a}}_y)e^{+j\beta z}E_a$

Determine the:

1. Polarization of the incident wave (linear, circular, elliptical; sense of rotation; and AR).
2. Polarization of antenna of part (a) (linear, circular, elliptical; sense of rotation; and AR).
3. Polarization of antenna of part (b) (linear, circular, elliptical; sense of rotation; and AR).
4. Normalized output voltage when the incident wave impinges upon the antenna whose electric field is that of part (a).
5. Normalized output voltage when the incident wave impinges upon the antenna whose electric field is that of part (b).

4.37. The field radiated by an antenna has electric field components represented by (4-50a) such that $E_{x_0}^+ = E_{y_0}^+$ and its Axial Ratio is infinity.

(a) Identify the polarization point on the Poincaré sphere (i.e., find γ, δ, τ, and ε in degrees).

(b) If this antenna is used to receive the wave of Problem 4.35, find the polarization loss (in decibels). To do this part, use the Poincaré sphere parameters.