CHAPTER 3

Wave Equation and its Solutions

3.1 INTRODUCTION

The electromagnetic fields of boundary-value problems are obtained as solutions to Maxwell's equations, which are first-order partial differential equations. However, Maxwell's equations are coupled partial differential equations, which means that each equation has more than one unknown field. These equations can be uncoupled only at the expense of raising their order. For each of the fields, following such a procedure leads to an uncoupled second-order partial differential equation that is usually referred to as the *wave equation*. Therefore electric and magnetic fields for a given boundary-value problem can be obtained either as solutions to Maxwell's or the wave equations. The choice of equations is related to individual problems by convenience and ease of use. In this chapter we will develop the vector wave equations for each of the fields, and then we will demonstrate their solutions in the rectangular, cylindrical, and spherical coordinate systems.

3.2 TIME-VARYING ELECTROMAGNETIC FIELDS

The first two of Maxwell's equations in differential form, as given by (1-1) and (1-2), are first-order, coupled differential equations; that is, both the unknown fields ($\boldsymbol{\mathscr{E}}$ and $\boldsymbol{\mathscr{H}}$) appear in each equation. Usually it is very desirable, for convenience in solving for $\boldsymbol{\mathscr{E}}$ and $\boldsymbol{\mathscr{H}}$, to uncouple these equations. This can be accomplished at the expense of increasing the order of the differential equations to second order. To do this, we repeat (1-1) and (1-2), that is,

$$\nabla \times \boldsymbol{\mathscr{E}} = -\boldsymbol{\mathscr{M}}_i - \mu \frac{\partial \boldsymbol{\mathscr{H}}}{\partial t} \tag{3-1}$$

$$\nabla \times \boldsymbol{\mathscr{H}} = \boldsymbol{\mathscr{J}}_i + \sigma \boldsymbol{\mathscr{E}} + \varepsilon \frac{\partial \boldsymbol{\mathscr{E}}}{\partial t} \tag{3-2}$$

where it is understood in the remaining part of the book that σ represents the effective conductivity σ_ε and ε represents ε'. Taking the curl of both sides of each of equations 3-1 and 3-2 and assuming a homogeneous medium, we can write that

$$\nabla \times \nabla \times \boldsymbol{\mathscr{E}} = -\nabla \times \boldsymbol{\mathscr{M}}_i - \mu \nabla \times \left(\frac{\partial \boldsymbol{\mathscr{H}}}{\partial t}\right) = -\nabla \times \boldsymbol{\mathscr{M}}_i - \mu \frac{\partial}{\partial t}(\nabla \times \boldsymbol{\mathscr{H}}) \tag{3-3}$$

$$\begin{aligned} \nabla \times \nabla \times \boldsymbol{\mathscr{H}} &= \nabla \times \boldsymbol{\mathscr{J}}_i + \sigma \nabla \times \boldsymbol{\mathscr{E}} + \varepsilon \nabla \times \left(\frac{\partial \boldsymbol{\mathscr{E}}}{\partial t}\right) \\ &= \nabla \times \boldsymbol{\mathscr{J}}_i + \sigma \nabla \times \boldsymbol{\mathscr{E}} + \varepsilon \frac{\partial}{\partial t}(\nabla \times \boldsymbol{\mathscr{E}}) \end{aligned} \tag{3-4}$$

Substituting (3-2) into the right side of (3-3) and using the vector identity

$$\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \tag{3-5}$$

into the left side, we can rewrite (3-3) as

$$\nabla(\nabla \cdot \boldsymbol{\mathscr{E}}) - \nabla^2 \boldsymbol{\mathscr{E}} = -\nabla \times \boldsymbol{\mathscr{M}}_i - \mu \frac{\partial}{\partial t}\left[\boldsymbol{\mathscr{J}}_i + \sigma \boldsymbol{\mathscr{E}} + \varepsilon \frac{\partial \boldsymbol{\mathscr{E}}}{\partial t}\right]$$

$$\nabla(\nabla \cdot \boldsymbol{\mathscr{E}}) - \nabla^2 \boldsymbol{\mathscr{E}} = -\nabla \times \boldsymbol{\mathscr{M}}_i - \mu \frac{\partial \boldsymbol{\mathscr{J}}_i}{\partial t} - \mu\sigma \frac{\partial \boldsymbol{\mathscr{E}}}{\partial t} - \mu\varepsilon \frac{\partial^2 \boldsymbol{\mathscr{E}}}{\partial t^2} \tag{3-6}$$

Substituting Maxwell's equation 1-3, or

$$\nabla \cdot \boldsymbol{\mathscr{D}} = \varepsilon \nabla \cdot \boldsymbol{\mathscr{E}} = \mathscr{q}_{ev} \Rightarrow \nabla \cdot \boldsymbol{\mathscr{E}} = \frac{\mathscr{q}_{ev}}{\varepsilon} \tag{3-7}$$

into (3-6) and rearranging its terms, we have that

$$\boxed{\nabla^2 \boldsymbol{\mathscr{E}} = \nabla \times \boldsymbol{\mathscr{M}}_i + \mu \frac{\partial \boldsymbol{\mathscr{J}}_i}{\partial t} + \frac{1}{\varepsilon}\nabla \mathscr{q}_{ev} + \mu\sigma \frac{\partial \boldsymbol{\mathscr{E}}}{\partial t} + \mu\varepsilon \frac{\partial^2 \boldsymbol{\mathscr{E}}}{\partial t^2}} \tag{3-8}$$

which is recognized as an uncoupled second-order differential equation for $\boldsymbol{\mathscr{E}}$.

In a similar manner, by substituting (3-1) into the right side of (3-4) and using the vector identity of (3-5) in the left side of (3-4), we can rewrite it as

$$\nabla(\nabla \cdot \boldsymbol{\mathscr{H}}) - \nabla^2 \boldsymbol{\mathscr{H}} = \nabla \times \boldsymbol{\mathscr{J}}_i + \sigma\left(-\boldsymbol{\mathscr{M}}_i - \mu \frac{\partial \boldsymbol{\mathscr{H}}}{\partial t}\right) + \varepsilon \frac{\partial}{\partial t}\left(-\boldsymbol{\mathscr{M}}_i - \mu \frac{\partial \boldsymbol{\mathscr{H}}}{\partial t}\right)$$

$$\nabla(\nabla \cdot \boldsymbol{\mathscr{H}}) - \nabla^2 \boldsymbol{\mathscr{H}} = \nabla \times \boldsymbol{\mathscr{J}}_i - \sigma \boldsymbol{\mathscr{M}}_i - \mu\sigma \frac{\partial \boldsymbol{\mathscr{H}}}{\partial t} - \varepsilon \frac{\partial \boldsymbol{\mathscr{M}}_i}{\partial t} - \mu\varepsilon \frac{\partial^2 \boldsymbol{\mathscr{H}}}{\partial t^2} \tag{3-9}$$

Substituting Maxwell's equation

$$\nabla \cdot \boldsymbol{\mathscr{B}} = \mu \nabla \cdot \boldsymbol{\mathscr{H}} = \mathscr{q}_{mv} \Rightarrow \nabla \cdot \boldsymbol{\mathscr{H}} = \left(\frac{\mathscr{q}_{mv}}{\mu}\right) \tag{3-10}$$

into (3-9), we have that

$$\boxed{\nabla^2 \boldsymbol{\mathscr{H}} = -\nabla \times \boldsymbol{\mathscr{J}}_i + \sigma \boldsymbol{\mathscr{M}}_i + \frac{1}{\mu}\nabla(\mathscr{q}_{mv}) + \varepsilon \frac{\partial \boldsymbol{\mathscr{M}}_i}{\partial t} + \mu\sigma \frac{\partial \boldsymbol{\mathscr{H}}}{\partial t} + \mu\varepsilon \frac{\partial^2 \boldsymbol{\mathscr{H}}}{\partial t^2}} \tag{3-11}$$

which is recognized as an uncoupled second-order differential equation for $\boldsymbol{\mathscr{H}}$. Thus (3-8) and (3-11) form a pair of uncoupled second-order differential equations that are a by-product of Maxwell's equations as given by (1-1) through (1-4).

Equations 3-8 and 3-11 are referred to as the *vector wave equations* for $\boldsymbol{\mathscr{E}}$ and $\boldsymbol{\mathscr{H}}$. For solving an electromagnetic boundary-value problem, the equations that must be satisfied are Maxwell's equations as given by (1-1) through (1-4) or the wave equations as given by (3-8) and (3-11). Often, the forms of the wave equations are preferred over those of Maxwell's equations.

For source-free regions ($\boldsymbol{\mathcal{J}}_i = \mathcal{q}_{ev} = 0$ and $\boldsymbol{\mathcal{M}}_i = \mathcal{q}_{mv} = 0$), the wave equations 3-8 and 3-11 reduce, respectively, to

$$\boxed{\nabla^2 \boldsymbol{\mathcal{E}} = \mu\sigma \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} + \mu\varepsilon \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2}} \tag{3-12}$$

$$\boxed{\nabla^2 \boldsymbol{\mathcal{H}} = \mu\sigma \frac{\partial \boldsymbol{\mathcal{H}}}{\partial t} + \mu\varepsilon \frac{\partial^2 \boldsymbol{\mathcal{H}}}{\partial t^2}} \tag{3-13}$$

For source-free ($\boldsymbol{\mathcal{J}}_i = \mathcal{q}_{ev} = 0$ and $\boldsymbol{\mathcal{M}}_i = \mathcal{q}_{mv} = 0$) and lossless media ($\sigma = 0$), the wave equations 3-8 and 3-11 or 3-12 and 3-13 simplify to

$$\boxed{\nabla^2 \boldsymbol{\mathcal{E}} = \mu\varepsilon \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2}} \tag{3-14}$$

$$\boxed{\nabla^2 \boldsymbol{\mathcal{H}} = \mu\varepsilon \frac{\partial^2 \boldsymbol{\mathcal{H}}}{\partial t^2}} \tag{3-15}$$

Equations 3-14 and 3-15 represent the simplest forms of the vector wave equations.

3.3 TIME-HARMONIC ELECTROMAGNETIC FIELDS

For time-harmonic fields (time variations of the form $e^{j\omega t}$), the wave equations can be derived using a similar procedure as in Section 3.2 for the general time-varying fields, starting with Maxwell's equations as given in Table 1-4. However, instead of going through this process, we find, by comparing Maxwell's equations for the general time-varying fields with those for the time-harmonic fields (both are displayed in Table 1-4), that one set can be obtained from the other by replacing $\partial/\partial t \equiv j\omega$, $\partial^2/\partial t^2 \equiv (j\omega)^2 = -\omega^2$, and the instantaneous fields ($\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{D}}, \boldsymbol{\mathcal{B}}$), respectively, with the complex fields ($\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}$) and vice versa. Doing this for the wave equations 3-8, 3-11, 3-12, and 3-13, we can write each, respectively, as

$$\nabla^2 \mathbf{E} = \nabla \times \mathbf{M}_i + j\omega\mu \mathbf{J}_i + \frac{1}{\varepsilon}\nabla q_{ev} + j\omega\mu\sigma \mathbf{E} - \omega^2 \mu\varepsilon \mathbf{E} \tag{3-16a}$$

$$\nabla^2 \mathbf{H} = -\nabla \times \mathbf{J}_i + \sigma \mathbf{M}_i + j\omega\varepsilon \mathbf{M}_i + \frac{1}{\mu}\nabla q_{mv} + j\omega\mu\sigma \mathbf{H} - \omega^2 \mu\varepsilon \mathbf{H} \tag{3-16b}$$

$$\nabla^2 \mathbf{E} = j\omega\mu\sigma \mathbf{E} - \omega^2 \mu\varepsilon \mathbf{E} = \gamma^2 \mathbf{E} \tag{3-17a}$$

$$\nabla^2 \mathbf{H} = j\omega\mu\sigma \mathbf{H} - \omega^2 \mu\varepsilon \mathbf{H} = \gamma^2 \mathbf{H} \tag{3-17b}$$

where

$$\gamma^2 = j\omega\mu\sigma - \omega^2\mu\varepsilon = j\omega\mu(\sigma + j\omega\varepsilon) \tag{3-17c}$$

$$\gamma = \alpha + j\beta = \text{propagation constant} \tag{3-17d}$$

$$\alpha = \text{attenuation constant (Np/m)} \tag{3-17e}$$

$$\beta = \text{phase constant (rad/m)} \tag{3-17f}$$

The constants α, β, and γ will be discussed in more detail in Section 4.3 where α and β are expressed by (4-28c) and (4-28d) in terms of ω, ε, μ, and σ.

Similarly (3-14) and (3-15) can be written, respectively, as

$$\boxed{\nabla^2\mathbf{E} = -\omega^2\mu\varepsilon\mathbf{E} = -\beta^2\mathbf{E}} \tag{3-18a}$$

$$\boxed{\nabla^2\mathbf{H} = -\omega^2\mu\varepsilon\mathbf{H} = -\beta^2\mathbf{H}} \tag{3-18b}$$

where

$$\beta^2 = \omega^2\mu\varepsilon \tag{3-18c}$$

In the literature the phase constant β is also represented by k.

3.4 SOLUTION TO THE WAVE EQUATION

The time variations of most practical problems are of the time-harmonic form. Fourier series can be used to express time variations of other forms in terms of a number of time-harmonic terms. Electromagnetic fields associated with a given boundary-value problem must satisfy Maxwell's equations or the vector wave equations. For many cases, the vector wave equations reduce to a number of scalar Helmholtz (wave) equations, and the general solutions can be constructed once solutions to each of the scalar Helmholtz equations are found.

In this section we want to demonstrate at least one method that can be used to solve the scalar Helmholtz equation in rectangular, cylindrical, and spherical coordinates. The method is known as the *separation of variables* [1, 2], and the general solution to the scalar Helmholtz equation using this method can be constructed in 11 three-dimensional orthogonal coordinate systems (including the rectangular, cylindrical, and spherical systems) [3].

The solutions for the instantaneous time-harmonic electric and magnetic field intensities can be obtained by considering the forms of the vector wave equations given either in Section 3.2 or Section 3.3. The approach chosen here will be to use those of Section 3.3 to solve for the complex field intensities $\mathbf{E}$ and $\mathbf{H}$ first. The corresponding instantaneous quantities can then be formed using the relations (1-61a) through (1-61f) between the instantaneous time-harmonic fields and their complex counterparts.

3.4.1 Rectangular Coordinate System

In a rectangular coordinate system, the vector wave equations 3-16a through 3-18c can be reduced to three scalar wave (Helmholtz) equations. First, we will consider the solutions for source-free and lossless media. This will be followed by solutions for source-free but lossy media.

A. Source-Free and Lossless Media For source-free ($\mathbf{J}_i = \mathbf{M}_i = q_{ve} = q_{vm} = 0$) and lossless ($\sigma = 0$) media, the vector wave equations for the complex electric and magnetic field intensities are those given by (3-18a) through (3-18c). Since (3-18a) and (3-18b) are of the same form, let us examine the solution to one of them. The solution to the other can then be written by an interchange of $\mathbf{E}$ with $\mathbf{H}$ or $\mathbf{H}$ with $\mathbf{E}$. We will begin by examining the solution for $\mathbf{E}$.

In rectangular coordinates, a general solution for $\mathbf{E}$ can be written as

$$\mathbf{E}(x, y, z) = \hat{\mathbf{a}}_x E_x(x, y, z) + \hat{\mathbf{a}}_y E_y(x, y, z) + \hat{\mathbf{a}}_z E_z(x, y, z) \tag{3-19}$$

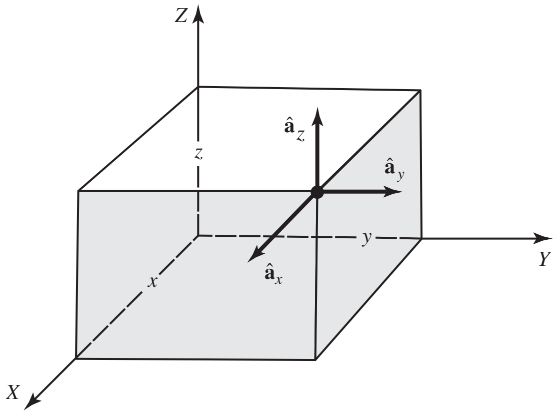

Figure 3-1 Rectangular coordinate system and corresponding unit vectors.

where x, y, z are the rectangular coordinates, as illustrated in Figure 3-1. Substituting (3-19) into (3-18a) we can write that

$$\nabla^2\mathbf{E} + \beta^2\mathbf{E} = \nabla^2\left(\hat{\mathbf{a}}_x E_x + \hat{\mathbf{a}}_y E_y + \hat{\mathbf{a}}_z E_z\right) + \beta^2\left(\hat{\mathbf{a}}_x E_x + \hat{\mathbf{a}}_y E_y + \hat{\mathbf{a}}_z E_z\right) = 0 \tag{3-20}$$

which reduces to three scalar wave equations of

$$\nabla^2 E_x(x, y, z) + \beta^2 E_x(x, y, z) = 0 \tag{3-20a}$$

$$\nabla^2 E_y(x, y, z) + \beta^2 E_y(x, y, z) = 0 \tag{3-20b}$$

$$\nabla^2 E_z(x, y, z) + \beta^2 E_z(x, y, z) = 0 \tag{3-20c}$$

because

$$\nabla^2\left(\hat{\mathbf{a}}_x E_x + \hat{\mathbf{a}}_y E_y + \hat{\mathbf{a}}_z E_z\right) = \hat{\mathbf{a}}_x \nabla^2 E_x + \hat{\mathbf{a}}_y \nabla^2 E_y + \hat{\mathbf{a}}_z \nabla^2 E_z \tag{3-21}$$

Equations 3-20a through 3-20c are all of the same form; once a solution of any one of them is obtained, the solutions to the others can be written by inspection. We choose to work first with that for E_x as given by (3-20a).

In expanded form (3-20a) can be written as

$$\nabla^2 E_x + \beta^2 E_x = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \beta^2 E_x = 0 \tag{3-22}$$

Using the *separation-of-variables method*, we assume that a solution for $E_x(x, y, z)$ can be written in the form of

$$E_x(x, y, z) = f(x)g(y)h(z) \tag{3-23}$$

where the x, y, z variations of E_x are separable (hence the name). If any inconsistencies are encountered with assuming such a form of solution, another form must be attempted. This is the procedure usually followed in solving differential equations. Substituting (3-23) into (3-22), we can write that

$$gh\frac{\partial^2 f}{\partial x^2} + fh\frac{\partial^2 g}{\partial y^2} + fg\frac{\partial^2 h}{\partial z^2} + \beta^2 fgh = 0 \tag{3-24}$$

Since $f(x)$, $g(y)$, and $h(z)$ are each a function of only one variable, we can replace the partials in (3-24) by ordinary derivatives. Doing this and dividing each term by fgh, we can write that

$$\frac{1}{f}\frac{d^2f}{dx^2}+\frac{1}{g}\frac{d^2g}{dy^2}+\frac{1}{h}\frac{d^2h}{dz^2}+\beta^2=0 \tag{3-25}$$

or

$$\frac{1}{f}\frac{d^2f}{dx^2}+\frac{1}{g}\frac{d^2g}{dy^2}+\frac{1}{h}\frac{d^2h}{dz^2}=-\beta^2 \tag{3-25a}$$

Each of the first three terms in (3-25a) is a function of only a single independent variable; hence the sum of these terms can equal $-\beta^2$ only if each term is a constant. Thus (3-25a) separates into three equations of the form

$$\frac{1}{f}\frac{d^2f}{dx^2}=-\beta_x^2 \Rightarrow \frac{d^2f}{dx^2}=-\beta_x^2 f \tag{3-26a}$$

$$\frac{1}{g}\frac{d^2g}{dy^2}=-\beta_y^2 \Rightarrow \frac{d^2g}{dy^2}=-\beta_y^2 g \tag{3-26b}$$

$$\frac{1}{h}\frac{d^2h}{dz^2}=-\beta_z^2 \Rightarrow \frac{d^2h}{dz^2}=-\beta_z^2 h \tag{3-26c}$$

where, in addition,

$$\beta_x^2+\beta_y^2+\beta_z^2=\beta^2 \tag{3-27}$$

Equation 3-27 is referred to as the *constraint (dispersion)* equation. In addition β_x, β_y, β_z are known as the wave constants (numbers) in the x, y, z directions, respectively, that will be determined using boundary conditions.

The solution to each of (3-26a), (3-26b), or (3-26c) can take different forms. Some typical valid solutions for $f(x)$ of (3-26a) would be

$$f_1(x)=A_1e^{-j\beta_x x}+B_1e^{+j\beta_x x} \tag{3-28a}$$

or

$$f_2(x)=C_1\cos(\beta_x x)+D_1\sin(\beta_x x) \tag{3-28b}$$

Similarly the solutions to (3-26b) and (3-26c) for $g(y)$ and $h(z)$ can be written, respectively, as

$$g_1(y)=A_2e^{-j\beta_y y}+B_2e^{+j\beta_y y} \tag{3-29a}$$

or

$$g_2(y)=C_2\cos(\beta_y y)+D_2\sin(\beta_y y) \tag{3-29b}$$

and

$$h_1(z)=A_3e^{-j\beta_z z}+B_3e^{+j\beta_z z} \tag{3-30a}$$

or

$$h_2(z)=C_3\cos(\beta_z z)+D_3\sin(\beta_z z) \tag{3-30b}$$

Although all the aforementioned solutions are valid for $f(x)$, $g(y)$, and $h(z)$, the most appropriate form should be chosen to simplify the complexity of the problem at hand. In general, the solutions of (3-28a), (3-29a), and (3-30a) in terms of complex exponentials represent *traveling waves* and the solutions of (3-28b), (3-29b), and (3-30b) represent *standing waves*. Wave functions representing various wave types in rectangular coordinates are found listed in Table 3-1. In

TABLE 3-1 Wave functions, zeroes, and infinities of plane wave functions in rectangular coordinates

Wave type	Wave functions	Zeroes of wave functions	Infinities of wave functions
Traveling waves	$e^{-j\beta x}$ for $+x$ travel $e^{+j\beta x}$ for $-x$ travel	$\beta x \to -j\infty$ $\beta x \to +j\infty$	$\beta x \to +j\infty$ $\beta x \to -j\infty$
Standing waves	$\cos(\beta x)$ for $\pm x$ $\sin(\beta x)$ for $\pm x$	$\beta x = \pm\left(n+\frac{1}{2}\right)\pi$ $\beta x = \pm n\pi$ $n = 0, 1, 2, \ldots$	$\beta x \to \pm j\infty$ $\beta x \to \pm j\infty$
Evanescent waves	$e^{-\alpha x}$ for $+x$ $e^{+\alpha x}$ for $-x$ $\cosh(\alpha x)$ for $\pm x$ $\sinh(\alpha x)$ for $\pm x$	$\alpha x \to +\infty$ $\alpha x \to -\infty$ $\alpha x = \pm j\left(n+\frac{1}{2}\right)\pi$ $\alpha x = \pm jn\pi$ $n = 0, 1, 2, \ldots$	$\alpha x \to -\infty$ $\alpha x \to +\infty$ $\alpha x \to \pm\infty$ $\alpha x \to \pm\infty$
Attenuating traveling waves	$e^{-\gamma x} = e^{-\alpha x}e^{-j\beta x}$ for $+x$ travel $e^{+\gamma x} = e^{+\alpha x}e^{+j\beta x}$ for $-x$ travel	$\gamma x \to +\infty$ $\gamma x \to -\infty$	$\gamma x \to -\infty$ $\gamma x \to +\infty$
Attenuating standing waves	$\cos(\gamma x) = \cos(\alpha x)\cosh(\beta x) - j\sin(\alpha x)\sinh(\beta x)$ for $\pm x$ $\sin(\gamma x) = \sin(\alpha x)\cosh(\beta x) + j\cos(\alpha x)\sinh(\beta x)$ for $\pm x$	$\gamma x = \pm j\left(n+\frac{1}{2}\right)\pi$ $\gamma x = \pm jn\pi$ $n = 0, 1, 2, \ldots$	$\gamma x \to \pm j\infty$ $\gamma x \to \pm j\infty$

Chapter 8 we will consider specific examples and the appropriate solution forms for $f(x)$, $g(y)$, and $h(z)$.

Once the appropriate forms for $f(x)$, $g(y)$, and $h(z)$ have been decided, the solution for the scalar function $E_x(x, y, z)$ of (3-22) can be written as the product of fgh as stated by (3-23). To demonstrate that, let us consider a specific example in which it will be assumed that the appropriate solutions for f, g, and h are given, respectively, by (3-28b), (3-29b), and (3-30a). Thus we can write that

$$E_x(x, y, z) = \left[C_1\cos(\beta_x x) + D_1\sin(\beta_x x)\right]\left[C_2\cos(\beta_y y) + D_2\sin(\beta_y y)\right] \times\left[A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right] \tag{3-31}$$

This is an appropriate solution for any of the electric or magnetic field components inside a rectangular pipe (waveguide), shown in Figure 3-2, that is bounded in the x and y directions and has its length along the z axis. Because the waveguide is bounded in the x and y directions, standing waves, represented by cosine and sine functions, have been chosen as solutions for $f(x)$ and $g(y)$ functions. However, because the waveguide is not bounded in the z direction, traveling waves, represented by complex exponential functions, have been chosen as solutions for $h(z)$. A complete discussion of the fields inside a rectangular waveguide can be found in Chapter 8.

For $e^{j\omega t}$ time variations, which are assumed throughout this book, the first complex exponential term in (3-31) represents a wave that travels in the $+z$ direction; the second exponential represents a wave that travels in the $-z$ direction. To demonstrate this, let us examine the instantaneous form

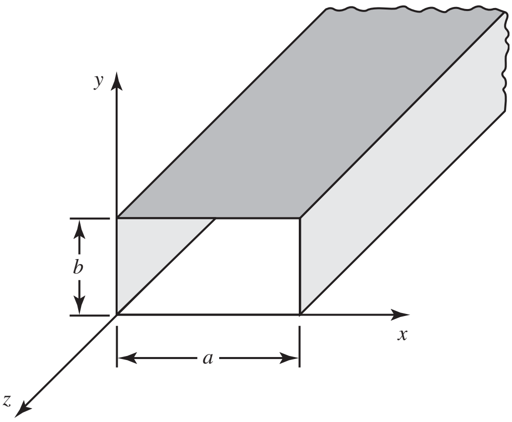

Figure 3-2 Rectangular waveguide geometry.

$\mathscr{E}_x(x, y, z; t)$ of the scalar complex function $E_x(x, y, z)$. Since the solution of (3-31) represents the complex form of E_x, its instantaneous form can be written as

$$\mathscr{E}_x(x, y, z; t) = \mathrm{Re}\left[E_x(x, y, z)e^{j\omega t}\right] \tag{3-32}$$

Considering only the first exponential term of (3-31) and assuming all constants are real, we can write the instantaneous form of the $\mathscr{E}_x$ function for that term as

$$\begin{aligned}\mathscr{E}_x^+(x, y, z; t) &= \mathrm{Re}\left[E_x^+(x, y, z)e^{j\omega t}\right] \\ &= \mathrm{Re}\Big\{\left[C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)\right] \\ &\qquad \times\left[C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)\right]A_3 e^{j(\omega t - \beta_z z)}\Big\}\end{aligned} \tag{3-33}$$

or, if the constants C_1, D_1, C_2, D_2, and A_3 are real, as

$$\begin{aligned}\mathscr{E}_x^+(x, y, z; t) &= \left[C_1 \cos(\beta_x x) + D_1 \sin(\beta_x x)\right] \\ &\qquad \times\left[C_2 \cos(\beta_y y) + D_2 \sin(\beta_y y)\right]A_3 \cos(\omega t - \beta_z z)\end{aligned} \tag{3-33a}$$

where the superscript plus is used to denote a positive traveling wave.

A plot of the normalized $\mathscr{E}_x^+(x, y, z; t)$ as a function of z for different times $(t = t_0, t_1, \ldots, t_n, t_{n+1})$ is shown in Figure 3-3. It is evident that as time increases $(t_{n+1} > t_n)$, the waveform of $\mathscr{E}_x^+$ is essentially the same, with the exception of an apparent shift in the $+z$ direction indicating a wave traveling in the $+z$ direction. This shift in the $+z$ direction can also be demonstrated by examining what happens to a given point z_p in the waveform of $\mathscr{E}_x^+$ for $t = t_0, t_1, \ldots, t_n, t_{n+1}$. To follow the point z_p for different values of t, we must maintain constant the amplitude of the last cosine term in (3-33a). This is accomplished by keeping its argument $\omega t - \beta_z z_p$ constant, that is,

$$\omega t - \beta_z z_p = C_0 = \text{constant} \tag{3-34}$$

which when differentiated with respect to time reduces to

$$\omega(1) - \beta_z \frac{dz_p}{dt} = 0 \Rightarrow \frac{dz_p}{dt} = v_p = +\frac{\omega}{\beta_z} \tag{3-35}$$

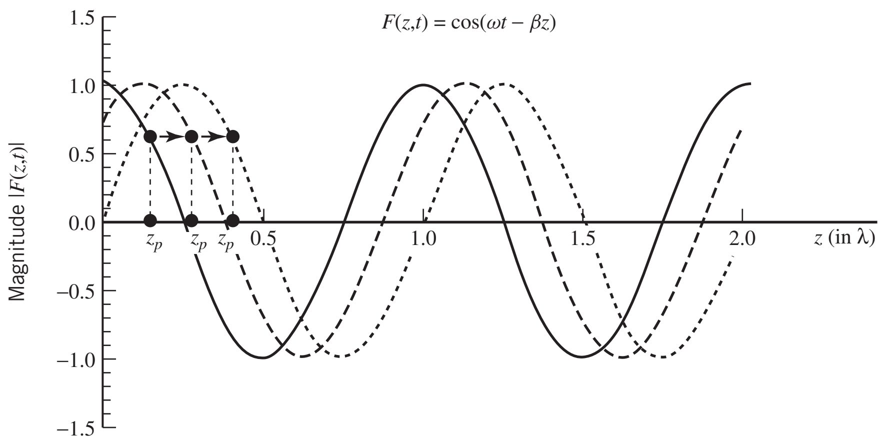

Figure 3-3 Variations as a function of distance for different times of positive traveling wave. —— time $t_0 = 0$; – – – – time $t_1 = T/8$; ---- time $t_2 = T/4$.

The point z_p is referred to as an *equiphase* point and its velocity is denoted as the *phase velocity*. A similar procedure can be used to demonstrate that the second complex exponential term in (3-31) represents a wave that travels in the $-z$ direction.

B. Source-Free and Lossy Media When the media in which the waves are traveling are lossy ($\sigma \neq 0$) but source-free ($\mathbf{J}_i = \mathbf{M}_i = q_{ve} = q_{vm} = 0$), the vector wave equations that the complex electric $\mathbf{E}$ and magnetic $\mathbf{H}$ field intensities must satisfy are (3-17a) and (3-17b). As for the lossless case, let us examine the solution to one of them; the solution to the other can then be written by inspection once the solution to the first has been obtained. We choose to consider the solution for the electric field intensity $\mathbf{E}$, which must satisfy (3-17a). An extended presentation of electromagnetic wave propagation in lossy media can be found in [4].

In a rectangular coordinate system, the general solution for $\mathbf{E}(x, y, z)$ can be written as

$$\mathbf{E}(x, y, z) = \hat{\mathbf{a}}_x E_x(x, y, z) + \hat{\mathbf{a}}_y E_y(x, y, z) + \hat{\mathbf{a}}_z E_z(x, y, z) \tag{3-36}$$

When (3-36) is substituted into (3-17a), we can write that

$$\nabla^2\mathbf{E} - \gamma^2\mathbf{E} = \nabla^2\left(\hat{\mathbf{a}}_x E_x + \hat{\mathbf{a}}_y E_y + \hat{\mathbf{a}}_z E_z\right) - \gamma^2\left(\hat{\mathbf{a}}_x E_x + \hat{\mathbf{a}}_y E_y + \hat{\mathbf{a}}_z E_z\right) = 0 \tag{3-37}$$

which reduces to three scalar wave equations of

$$\nabla^2 E_x(x, y, z) - \gamma^2 E_x(x, y, z) = 0 \tag{3-37a}$$

$$\nabla^2 E_y(x, y, z) - \gamma^2 E_y(x, y, z) = 0 \tag{3-37b}$$

$$\nabla^2 E_z(x, y, z) - \gamma^2 E_z(x, y, z) = 0 \tag{3-37c}$$

where

$$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) \tag{3-37d}$$

If we were to allow for positive and negative values of σ

$$\gamma = \pm\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \begin{cases} \pm(\alpha + j\beta) & \text{for} + \sigma \\ \pm(\alpha - j\beta) & \text{for} - \sigma \end{cases} \tag{3-37e}$$

In (3-37e),

$$\gamma = \text{propagation constant}$$
$$\alpha = \text{attenuation constant (Np/m)}$$
$$\beta = \text{phase constant (rad/m)}$$

where α and β are assumed to be real and positive. Although some authors choose to represent the phase constant by k, the symbol β will be used throughout this book.

Examining (3-37e) reveals that there are four possible combinations for the form of γ. That is,

$$\gamma = \begin{cases} +(\alpha + j\beta) & \text{(3-38a)} \\ -(\alpha + j\beta) & \text{(3-38b)} \\ +(\alpha - j\beta) & \text{(3-38c)} \\ -(\alpha - j\beta) & \text{(3-38d)} \end{cases}$$

Of the four combinations, only one will be appropriate for our solution. That form will be selected once the solutions to any of (3-37a) through (3-37c) have been decided.

Since all three equations represented by (3-37a) through (3-37c) are of the same form, let us examine only one of them. We choose to work first with (3-37a) whose solution can be derived using the method of *separation of variables*. Using a similar procedure as for the lossless case, we can write that

$$E_x(x, y, z) = f(x)g(y)h(z) \tag{3-39}$$

where it can be shown that $f(x)$ has solutions of the form

$$f_1(x) = A_1 e^{-\gamma_x x} + B_1 e^{+\gamma_x x} \tag{3-40a}$$

or

$$f_2(x) = C_1 \cosh(\gamma_x x) + D_1 \sinh(\gamma_x x) \tag{3-40b}$$

and $g(y)$ can be expressed as

$$g_1(y) = A_2 e^{-\gamma_y y} + B_2 e^{+\gamma_y y} \tag{3-41a}$$

or

$$g_2(y) = C_2 \cosh(\gamma_y y) + D_2 \sinh(\gamma_y y) \tag{3-41b}$$

and $h(z)$ as

$$h_1(z) = A_3 e^{-\gamma_z z} + B_3 e^{+\gamma_z z} \tag{3-42a}$$

or

$$h_2(z) = C_3 \cosh(\gamma_z z) + D_3 \sinh(\gamma_z z) \tag{3-42b}$$

Whereas (3-40a) through (3-42b) are appropriate solutions for f, g, and h of (3-39), which satisfy (3-37a), the *constraint* (*dispersion*) *equation* takes the form of

$$\boxed{\gamma_x^2 + \gamma_y^2 + \gamma_z^2 = \gamma^2} \tag{3-43}$$

The appropriate forms of f, g, and h chosen to represent the solution of $E_x(x, y, z)$, as given by (3-39), must be made by examining the geometry of the problem in question. As for the lossless case, the exponentials represent attenuating traveling waves and the hyperbolic cosines and sines represent attenuating standing waves. These and other waves types are listed in Table 3-1.

To decide on the appropriate form for any of the γ's (whether it be γ_x, γ_y, γ_z, or γ), let us choose the form of γ_z by examining one of the exponentials in (3-42a). We choose to work with the first one. The four possible combinations for γ_z, according to (3-38a) through (3-38d) will be

$$\gamma_z = \begin{cases} +(\alpha_z + j\beta_z) & \text{(3-44a)} \\ -(\alpha_z + j\beta_z) & \text{(3-44b)} \\ +(\alpha_z - j\beta_z) & \text{(3-44c)} \\ -(\alpha_z - j\beta_z) & \text{(3-44d)} \end{cases}$$

If we want the first exponential in (3-42a) to represent a decaying wave which travels in the $+z$ direction, then by substituting (3-44a) through (3-44d) into it we can write that

$$h_1^+(z) = \begin{cases} A_3 e^{-\gamma_z z} = A_3 e^{-\alpha_z z} e^{-j\beta_z z} & \text{(3-45a)} \\ A_3 e^{-\gamma_z z} = A_3 e^{+\alpha_z z} e^{+j\beta_z z} & \text{(3-45b)} \\ A_3 e^{-\gamma_z z} = A_3 e^{-\alpha_z z} e^{+j\beta_z z} & \text{(3-45c)} \\ A_3 e^{-\gamma_z z} = A_3 e^{+\alpha_z z} e^{-j\beta_z z} & \text{(3-45d)} \end{cases}$$

By examining (3-45a) through (3-45d) and assuming $e^{j\omega t}$ time variations, the following statements can be made:

1. Equation 3-45a represents a wave that travels in the $+z$ direction, as determined by $e^{-j\beta_z z}$, and it decays in that direction, as determined by $e^{-\alpha_z z}$.
2. Equation 3-45b represents a wave that travels in the $-z$ direction, as determined by $e^{+j\beta_z z}$, and it decays in that direction, as determined by $e^{+\alpha_z z}$.
3. Equation 3-45c represents a wave that travels in the $-z$ direction, as determined by $e^{+j\beta_z z}$, and it is increasing in that direction, as determined by $e^{-\alpha_z z}$.
4. Equation 3-45d represents a wave that travels in the $+z$ direction, as determined by $e^{-j\beta_z z}$, and it is increasing in that direction, as determined by $e^{+\alpha_z z}$.

From the preceding statements it is apparent that for $e^{-\gamma_z z}$ to represent a wave that travels in the $+z$ direction and that concurrently also decays (to represent propagation in passive lossy media), and to satisfy the conservation of energy laws, the only correct form of γ_z is that of (3-44a). The same conclusion will result if the second exponential of (3-42a) represents a wave that travels in the $-z$ direction and that concurrently also decays. Thus the general form of any γ_i (whether it be γ_x, γ_y, γ_z, or γ), as given by (3-38a) through (3-38d), is

$$\gamma_i = \alpha_i + j\beta_i \tag{3-46}$$

Whereas the forms of f, g, and h [as given by (3-40a) through (3-42b)] are used to arrive at the solution for the complex form of E_x as given by (3-39), the instantaneous form of $\mathscr{E}_x$ can be obtained by using the relation of (3-32). A similar procedure can be used to derive the solutions of the other components of $\mathbf{E}$ (E_y and E_z), all those of $\mathbf{H}$ (H_x, H_y, and H_z), and of their instantaneous counterparts.

3.4.2 Cylindrical Coordinate System

If the geometry of the system is of a cylindrical configuration, it would be very advisable to solve the boundary-value problem for the **E** and **H** fields using cylindrical coordinates. Maxwell's equations and the vector wave equations, which the **E** and **H** fields must satisfy, should be solved using cylindrical coordinates. Let us first consider the solution for **E** for a source-free and lossless medium. A similar procedure can be used for **H**. To maintain some simplicity in the mathematics, we will examine only lossless media.

In cylindrical coordinates a general solution to the vector wave equation for source-free and lossless media, as given by (3-18a), can be written as

$$\mathbf{E}(\rho,\phi,z) = \hat{\mathbf{a}}_\rho E_\rho(\rho,\phi,z) + \hat{\mathbf{a}}_\phi E_\phi(\rho,\phi,z) + \hat{\mathbf{a}}_z E_z(\rho,\phi,z) \tag{3-47}$$

where ρ, ϕ, and z are the cylindrical coordinates as illustrated in Figure 3-4. Substituting (3-47) into (3-18a), we can write that

$$\nabla^2(\hat{\mathbf{a}}_\rho E_\rho + \hat{\mathbf{a}}_\phi E_\phi + \hat{\mathbf{a}}_z E_z) = -\beta^2(\hat{\mathbf{a}}_\rho E_\rho + \hat{\mathbf{a}}_\phi E_\phi + \hat{\mathbf{a}}_z E_z) \tag{3-48}$$

which does not reduce to three simple scalar wave equations, similar to those of (3-20a) through (3-20c) for (3-20), because

$$\nabla^2(\hat{\mathbf{a}}_\rho E_\rho) \neq \hat{\mathbf{a}}_\rho \nabla^2 E_\rho \tag{3-49a}$$

$$\nabla^2(\hat{\mathbf{a}}_\phi E_\phi) \neq \hat{\mathbf{a}}_\phi \nabla^2 E_\phi \tag{3-49b}$$

However, because

$$\nabla^2(\hat{\mathbf{a}}_z E_z) = \hat{\mathbf{a}}_z \nabla^2 E_z \tag{3-49c}$$

one of the three scalar equations to which (3-48) reduces is

$$\nabla^2 E_z + \beta^2 E_z = 0 \tag{3-50}$$

The other two are of more complex form and they will be addressed in what follows.

Before we derive the other two scalar equations [in addition to (3-50)] to which (3-48) reduces, let us attempt to give a physical explanation of (3-49a), (3-49b), and (3-49c). By examining two different points (ρ_1, ϕ_1, z_1) and (ρ_2, ϕ_2, z_2) and their corresponding unit vectors on a cylindrical surface (as shown in Figure 3-4), we see that the directions of $\hat{\mathbf{a}}_\rho$ and $\hat{\mathbf{a}}_\phi$ have changed from one point to another (they are not parallel) and therefore cannot be treated as constants but rather are functions of ρ, ϕ, and z. In contrast, the unit vector $\hat{\mathbf{a}}_z$ at the two points is pointed in the same direction (is parallel). The same is true for the unit vectors $\hat{\mathbf{a}}_x$ and $\hat{\mathbf{a}}_y$ in Figure 3-1.

Let us now return to the solution of (3-48). Since (3-48) does not reduce to (3-49a) and (3-49b), although it does satisfy (3-49c), how do we solve (3-48)? The procedure that follows can be used to reduce (3-48) to three scalar partial differential equations.

The form of (3-48) written in general as

$$\nabla^2\mathbf{E} = -\beta^2\mathbf{E} \tag{3-51}$$

was placed in this form by utilizing the vector identity of (3-5) during its derivation. Generally we are under the impression that we do not know how to perform the Laplacian of a vector ($\nabla^2\mathbf{E}$) as given by the left side of (3-51). However, by utilizing (3-5) we can rewrite the left side of (3-51) as

$$\nabla^2\mathbf{E} = \nabla(\nabla\cdot\mathbf{E}) - \nabla\times\nabla\times\mathbf{E} \tag{3-52}$$

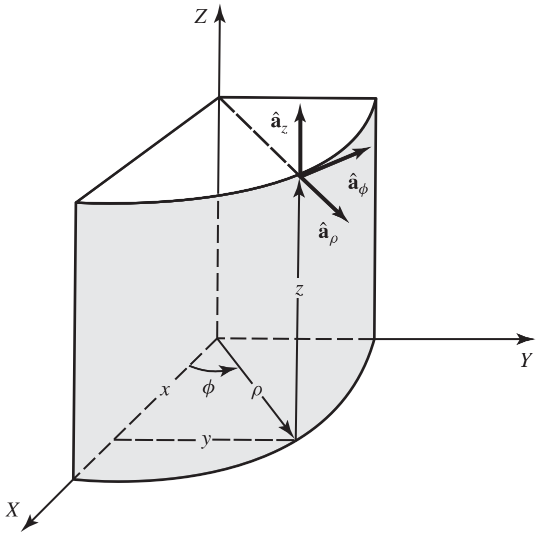

Figure 3-4 Cylindrical coordinate system and corresponding unit vectors.

whose terms can be expanded in any coordinate system. Using (3-52) we can write (3-51) as

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla \times \nabla \times \mathbf{E} = -\beta^2 \mathbf{E} \tag{3-53}$$

which is an alternate form, but not as commonly recognizable, of the vector wave equation for the electric field in source-free and lossless media.

Assuming a solution for the electric field of the form given by (3-47), we can expand (3-53) and reduce it to three scalar partial differential equations of the form

$$\nabla^2 E_\rho + \left(-\frac{E_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial E_\phi}{\partial \phi}\right) = -\beta^2 E_\rho \tag{3-54a}$$

$$\nabla^2 E_\phi + \left(-\frac{E_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial E_\rho}{\partial \phi}\right) = -\beta^2 E_\phi \tag{3-54b}$$

$$\nabla^2 E_z = -\beta^2 E_z \tag{3-54c}$$

In each of (3-54a) through (3-54c) $\nabla^2\psi(\rho, \phi, z)$ is the Laplacian of a scalar that in cylindrical coordinates takes the form of

$$\begin{aligned}\nabla^2\psi(\rho, \phi, z) &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \\ &= \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}\end{aligned} \tag{3-55}$$

Equations 3-54a and 3-54b are *coupled* (each contains more than one electric field component) second-order partial differential equations, which are the most difficult to solve. However, (3-54c) is an *uncoupled* second-order partial differential equation whose solution will be most useful in the construction of TE^z and TM^z mode solutions of boundary-value problems, as discussed in Chapters 6 and 9.

In expanded form (3-54c) can then be written as

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} = -\beta^2 \psi \tag{3-56}$$

where $\psi(\rho, \phi, z)$ is a scalar function that can represent a field or a vector potential component. Assuming a separable solution for $\psi(\rho, \phi, z)$ of the form

$$\psi(\rho, \phi, z) = f(\rho)g(\phi)h(z) \tag{3-57}$$

and substituting it into (3-56), we can write that

$$gh\frac{\partial^2 f}{\partial \rho^2} + gh\frac{1}{\rho}\frac{\partial f}{\partial \rho} + fh\frac{1}{\rho^2}\frac{\partial^2 g}{\partial \phi^2} + fg\frac{\partial^2 h}{\partial z^2} = -\beta^2 fgh \tag{3-58}$$

Dividing both sides of (3-58) by fgh and replacing the partials by ordinary derivatives reduces (3-58) to

$$\frac{1}{f}\frac{d^2 f}{d\rho^2} + \frac{1}{f}\frac{1}{\rho}\frac{df}{d\rho} + \frac{1}{g}\frac{1}{\rho^2}\frac{d^2 g}{d\phi^2} + \frac{1}{h}\frac{d^2 h}{dz^2} = -\beta^2 \tag{3-59}$$

The last term on the left side of (3-59) is only a function of z. Therefore, using the discussion of Section 3.4.1, we can write that

$$\frac{1}{h}\frac{d^2 h}{dz^2} = -\beta_z^2 \Rightarrow \frac{d^2 h}{dz^2} = -\beta_z^2 h \tag{3-60}$$

where β_z is a constant. Substituting (3-60) into (3-59) and multiplying both sides by ρ^2, reduces it to

$$\frac{\rho^2}{f}\frac{d^2 f}{d\rho^2} + \frac{\rho}{f}\frac{df}{d\rho} + \frac{1}{g}\frac{d^2 g}{d\phi^2} + (\beta^2 - \beta_z^2)\rho^2 = 0 \tag{3-61}$$

Since the third term on the left side of (3-61) is only a function of ϕ, it can be set equal to a constant $-m^2$. Thus we can write that

$$\frac{1}{g}\frac{d^2 g}{d\phi^2} = -m^2 \Rightarrow \frac{d^2 g}{d\phi^2} = -m^2 g \tag{3-62}$$

Letting

$$\beta^2 - \beta_z^2 = \beta_\rho^2 \Rightarrow \beta_\rho^2 + \beta_z^2 = \beta^2 \tag{3-63}$$

then using (3-62), and multiplying both sides of (3-61) by f, we can reduce (3-61) to

$$\rho^2\frac{d^2 f}{d\rho^2} + \rho\frac{df}{d\rho} + \left[(\beta_\rho \rho)^2 - m^2\right]f = 0 \tag{3-64}$$

Equation 3-63 is referred to as the *constraint (dispersion)* equation for the solution to the wave equation in cylindrical coordinates, and (3-64) is recognized as the classic *Bessel differential equation* [1–3, 5–10].

In summary then, the partial differential equation 3-56 whose solution was assumed to be separable of the form given by (3-57) reduces to the three differential equations 3-60, 3-62, 3-64 and the constraint equation 3-63. Thus

$$\nabla^2\psi(\rho,\phi,z) = \frac{\partial^2\psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} = -\beta^2\psi \tag{3-65}$$

where

$$\psi(\rho,\phi,z) = f(\rho)g(\phi)h(z) \tag{3-65a}$$

reduces to

$$\boxed{\rho^2\frac{d^2f}{d\rho^2} + \rho\frac{df}{d\rho} + \left[(\beta_\rho\rho)^2 - m^2\right]f = 0} \tag{3-66a}$$

$$\boxed{\frac{d^2g}{d\phi^2} = -m^2 g} \tag{3-66b}$$

$$\boxed{\frac{d^2h}{dz^2} = -\beta_z^2 h} \tag{3-66c}$$

with

$$\boxed{\beta_\rho^2 + \beta_z^2 = \beta^2} \tag{3-66d}$$

Solutions to (3-66a), (3-66b), and (3-66c) take the form, respectively, of

$$f_1(\rho) = A_1 J_m(\beta_\rho\rho) + B_1 Y_m(\beta_\rho\rho) \tag{3-67a}$$

or

$$f_2(\rho) = C_1 H_m^{(1)}(\beta_\rho\rho) + D_1 H_m^{(2)}(\beta_\rho\rho) \tag{3-67b}$$

and

$$g_1(\phi) = A_2 e^{-jm\phi} + B_2 e^{+jm\phi} \tag{3-68a}$$

or

$$g_2(\phi) = C_2\cos(m\phi) + D_2\sin(m\phi) \tag{3-68b}$$

and

$$h_1(z) = A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z} \tag{3-69a}$$

or

$$h_2(z) = C_3\cos(\beta_z z) + D_3\sin(\beta_z z) \tag{3-69b}$$

In (3-67a) $J_m(\beta_\rho\rho)$ and $Y_m(\beta_\rho\rho)$ represent, respectively, the Bessel functions of the first and second kind; $H_m^{(1)}(\beta_\rho\rho)$ and $H_m^{(2)}(\beta_\rho\rho)$ in (3-67b) represent, respectively, the Hankel functions of the first and second kind. A more detailed discussion of Bessel and Hankel functions is found in Appendix IV.

Although (3-67a) through (3-69b) are valid solutions for $f(\rho)$, $g(\phi)$, and $h(z)$, the most appropriate form will depend on the problem in question. For example, for the cylindrical waveguide of

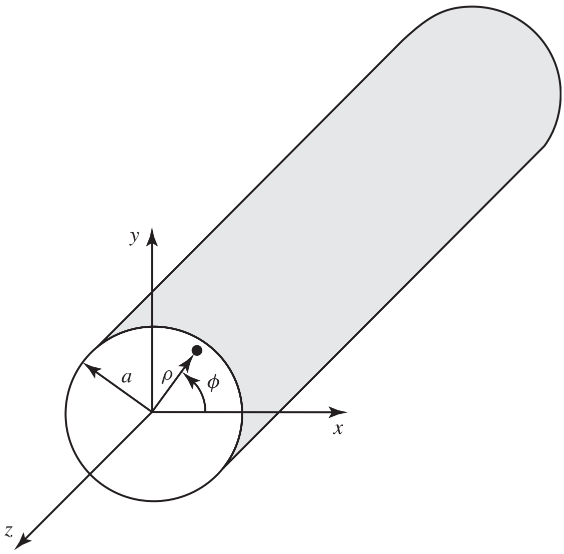

Figure 3-5 Cylindrical waveguide of the circular cross section.

Figure 3-5 the most convenient solutions for $f(\rho)$, $g(\phi)$, and $h(z)$ are those given, respectively, by (3-67a), (3-68b), and (3-69a). Thus we can write

$$\begin{aligned}\psi_1(\rho,\phi,z) &= f(\rho)g(\phi)h(z)\\ &= \left[A_1 J_m(\beta_\rho \rho) + B_1 Y_m(\beta_\rho \rho)\right]\\ &\quad \times [C_2\cos(m\phi) + D_2\sin(m\phi)]\left[A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right]\end{aligned} \tag{3-70}$$

These forms for $f(\rho)$, $g(\phi)$, and $h(z)$ were chosen in cylindrical coordinates for the following reasons.

1. Bessel functions of (3-67a) are used to represent standing waves, whereas Hankel functions of (3-67b) represent traveling waves.
2. Exponentials of (3-68a) represent traveling waves, whereas the cosines and sines of (3-68b) represent periodic waves.
3. Exponentials of (3-69a) represent traveling waves, whereas the cosines and sines of (3-69b) represent standing waves.

Wave functions representing various radial waves in cylindrical coordinates are found listed in Table 3-2.

Within the circular waveguide of Figure 3-5 standing waves are created in the radial (ρ) direction, periodic waves in the phi (ϕ) direction, and traveling waves in the z direction. For the fields to be finite at $\rho = 0$, where $Y_m(\beta_\rho \rho)$ possesses a singularity, (3-70) reduces to

$$\psi_1(\rho,\phi,z) = A_1 J_m(\beta_\rho \rho)[C_2\cos(m\phi) + D_2\sin(m\phi)]\left[A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right] \tag{3-70a}$$

To represent the fields in the region outside the cylinder, like scattering by the cylinder, a typical solution for $\psi(\rho,\phi,z)$ would take the form of

$$\psi_2(\rho,\phi,z) = B_1 H_m^{(2)}(\beta_\rho \rho)[C_2\cos(m\phi) + D_2\sin(m\phi)]\left[A_3 e^{-j\beta_z z} + B_3 e^{+j\beta_z z}\right] \tag{3-70b}$$

TABLE 3-2 Wave functions, zeroes, and infinities for radial wave functions in cylindrical coordinates

Wave type	Wave functions	Zeroes of wave functions	Infinities of wave functions
Traveling waves	$H_m^{(1)}(\beta\rho) = J_m(\beta\rho) + jY_m(\beta\rho)$ for $-\rho$ travel	$\beta\rho \to +j\infty$	$\beta\rho = 0$ $\beta\rho \to -j\infty$
	$H_m^{(2)}(\beta\rho) = J_m(\beta\rho) - jY_m(\beta\rho)$ for $+\rho$ travel	$\beta\rho \to -j\infty$	$\beta\rho = 0$ $\beta\rho \to +j\infty$
Standing waves	$J_m(\beta\rho)$ for $\pm\,\rho$	Infinite number (see Table 9-2)	$\beta\rho \to \pm j\infty$
	$Y_m(\beta\rho)$ for $\pm\,\rho$	Infinite number	$\beta\rho = 0$ $\beta\rho \to \pm j\infty$
Evanescent waves	$K_m(\alpha\rho) = \frac{\pi}{2}(-j)^{m+1}H_m^{(2)}(-j\alpha\rho)$ for $+\,\rho$	$\alpha\rho \to +\infty$	$\alpha\rho \to 0$ $\alpha\rho \to +\infty$ for integer orders
	$I_m(\alpha\rho) = j^m J_m(-j\alpha\rho)$ for $-\,\rho$		
Attenuating traveling waves	$H_m^{(1)}(-j\gamma\rho) = H_m^{(1)}(-j\alpha\rho + \beta\rho)$ for $-\,\rho$ travel	$\gamma\rho \to -\infty$	$\gamma\rho \to +\infty$
	$H_m^{(2)}(-j\gamma\rho) = H_m^{(2)}(-j\alpha\rho + \beta\rho)$ for $+\,\rho$ travel	$\gamma\rho \to +\infty$	$\gamma\rho \to -\infty$
Attenuating standing waves	$J_m(-j\gamma\rho) = J_m(-j\alpha\rho + \beta\rho)$ for $\pm\rho$	Infinite number	$\gamma\rho \to \pm j\infty$
	$Y_m(-j\gamma\rho) = Y_m(-j\alpha\rho + \beta\rho)$ for $\pm\,\rho$	Infinite number	$\gamma\rho \to \pm j\infty$

whereby the Hankel function of the second kind $H_m^{(2)}(\beta_\rho\rho)$ has replaced the Bessel function of the first kind $J_m(\beta_\rho\rho)$ because outward traveling waves are formed outside the cylinder, in contrast to the standing waves inside the cylinder.

More details concerning the application and properties of Bessel and Hankel function can be found in Chapters 9 and 11.

3.4.3 Spherical Coordinate System

Spherical coordinates should be utilized in solving problems that exhibit spherical geometries. As for the rectangular and cylindrical geometries, the electric and magnetic fields of a spherical geometry boundary-value problem must satisfy the corresponding vector wave equation, which is most conveniently solved in spherical coordinates as illustrated in Figure 3-6.

To simplify the problem, let us assume that the space in which the electric and magnetic fields must be solved is source-free and lossless. A general solution for the electric field can then be written as

$$\mathbf{E}(r,\theta,\phi) = \hat{\mathbf{a}}_r E_r(r,\theta,\phi) + \hat{\mathbf{a}}_\theta E_\theta(r,\theta,\phi) + \hat{\mathbf{a}}_\phi E_\phi(r,\theta,\phi) \tag{3-71}$$

Substituting (3-71) into the vector wave equation of (3-18a), we can write that

$$\nabla^2(\hat{\mathbf{a}}_r E_r + \hat{\mathbf{a}}_\theta E_\theta + \hat{\mathbf{a}}_\phi E_\phi) = -\beta^2(\hat{\mathbf{a}}_r E_r + \hat{\mathbf{a}}_\theta E_\theta + \hat{\mathbf{a}}_\phi E_\phi) \tag{3-72}$$

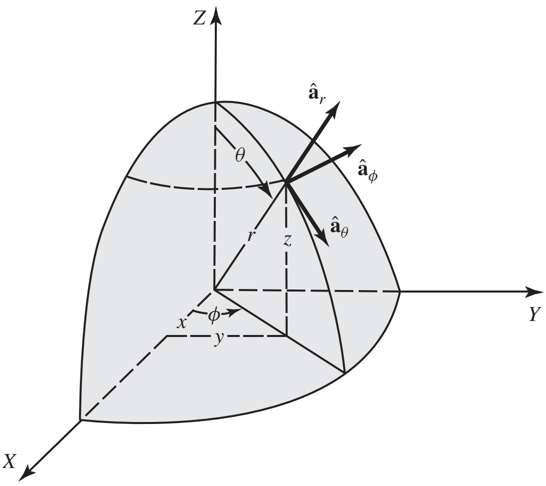

Figure 3-6 Spherical coordinate system and corresponding unit vectors.

Since

$$\nabla^2(\hat{\mathbf{a}}_r E_r) \neq \hat{\mathbf{a}}_r \nabla^2 E_r \tag{3-73a}$$

$$\nabla^2(\hat{\mathbf{a}}_\theta E_\theta) \neq \hat{\mathbf{a}}_\theta \nabla^2 E_\theta \tag{3-73b}$$

$$\nabla^2(\hat{\mathbf{a}}_\phi E_\phi) \neq \hat{\mathbf{a}}_\phi \nabla^2 E_\phi \tag{3-73c}$$

(3-72) does not reduce to three simple scalar wave equations, similar to those of (3-20a) through (3-20c) for (3-20). Therefore the reduction of (3-72) to three scalar partial differential equations must proceed in a different manner. In fact, the method used here will be similar to that utilized in cylindrical coordinates to reduce the vector wave equation to three scalar partial differential equations.

To accomplish this, we first rewrite the vector wave equation of (3-51) in a form given by (3-53) where now all the operators on the left side can be performed in any coordinate system. Substituting (3-71) into (3-53) shows that, after some lengthy mathematical manipulations, (3-53) reduces to three scalar partial differential equations of the form

$$\nabla^2 E_r - \frac{2}{r^2}\left(E_r + E_\theta \cot\theta + \csc\theta \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_\theta}{\partial \theta}\right) = -\beta^2 E_r \tag{3-74a}$$

$$\nabla^2 E_\theta - \frac{1}{r^2}\left(E_\theta \csc^2\theta - 2\frac{\partial E_r}{\partial \theta} + 2\cot\theta \csc\theta \frac{\partial E_\phi}{\partial \phi}\right) = -\beta^2 E_\theta \tag{3-74b}$$

$$\nabla^2 E_\phi - \frac{1}{r^2}\left(E_\phi \csc^2\theta - 2\csc\theta \frac{\partial E_r}{\partial \phi} - 2\cot\theta \csc\theta \frac{\partial E_\theta}{\partial \phi}\right) = -\beta^2 E_\phi \tag{3-74c}$$

Unfortunately, all three of the preceding partial differential equations are coupled. This means each contains more than one component of the electric field and would be most difficult to solve in its present form. However, as will be shown in Chapter 10, TE^r and TM^r wave mode solutions can be formed that in spherical coordinates must satisfy the scalar wave equation of

$$\nabla^2 \psi(r,\theta,\phi) = -\beta^2 \psi(r,\theta,\phi) \tag{3-75}$$

where $\psi(r,\theta,\phi)$ is a scalar function that can represent a field or a vector potential component. Therefore, it would be advisable here to demonstrate the solution to (3-75) in spherical coordinates.

Assuming a separable solution for $\psi(r,\theta,\phi)$ of the form

$$\psi(r,\theta,\phi) = f(r)g(\theta)h(\phi) \tag{3-76}$$

we can write the expanded form of (3-75)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial\psi}{\partial r}\right\} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial\psi}{\partial\theta}\right\} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = -\beta^2\psi \tag{3-77}$$

as

$$gh\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial f}{\partial r}\right\} + fh\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial g}{\partial\theta}\right\} + fg\frac{1}{r^2\sin^2\theta}\frac{\partial^2 h}{\partial\phi^2} = -\beta^2 fgh \tag{3-78}$$

Dividing both sides by fgh, multiplying by $r^2\sin^2\theta$, and replacing the partials by ordinary derivatives reduces (3-78) to

$$\frac{\sin^2\theta}{f}\frac{d}{dr}\left\{r^2\frac{df}{dr}\right\} + \frac{\sin\theta}{g}\frac{d}{d\theta}\left\{\sin\theta\frac{dg}{d\theta}\right\} + \frac{1}{h}\frac{d^2h}{d\phi^2} = -(\beta r\sin\theta)^2 \tag{3-79}$$

Since the last term on the left side of (3-79) is only a function of ϕ, it can be set equal to

$$\frac{1}{h}\frac{d^2h}{d\phi^2} = -m^2 \Rightarrow \frac{d^2h}{d\phi^2} = -m^2 h \tag{3-80}$$

where m is a constant.

Substituting (3-80) into (3-79), dividing both sides by $\sin^2\theta$, and transposing the term from the right to the left side reduces (3-79) to

$$\frac{1}{f}\frac{d}{dr}\left\{r^2\frac{df}{dr}\right\} + (\beta r)^2 + \frac{1}{g\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{dg}{d\theta}\right\} - \left\{\frac{m}{\sin\theta}\right\}^2 = 0 \tag{3-81}$$

Since the last two terms on the left side of (3-81) are only a function of θ, we can set them equal to

$$\frac{1}{g\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{dg}{d\theta}\right\} - \left\{\frac{m}{\sin\theta}\right\}^2 = -n(n+1) \tag{3-82}$$

where n is usually an integer. Equation 3-82 is closely related to the well-known *Legendre differential equation* (see Appendix V) [1–3, 6–10].

Substituting (3-82) into (3-81) reduces it to

$$\frac{1}{f}\frac{d}{dr}\left\{r^2\frac{df}{dr}\right\} + (\beta r)^2 - n(n+1) = 0 \tag{3-83}$$

which is closely related to the Bessel differential equation (see Appendix IV).

In summary then, the scalar wave equation 3-75 whose expanded form in spherical coordinates can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial\psi}{\partial r}\right\} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial\psi}{\partial\theta}\right\} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = -\beta^2\psi \tag{3-84}$$

and whose separable solution takes the form of

$$\boxed{\psi(r,\theta,\phi) = f(r)g(\theta)h(\phi)} \tag{3-85}$$

reduces to the three scalar differential equations

$$\boxed{\frac{d}{dr}\left\{r^2\frac{df}{dr}\right\} + \left[(\beta r)^2 - n(n+1)\right]f = 0} \tag{3-86a}$$

$$\boxed{\frac{1}{\sin\theta}\frac{d}{d\theta}\left\{\sin\theta\frac{dg}{d\theta}\right\} + \left[n(n+1) - \left\{\frac{m}{\sin\theta}\right\}^2\right]g = 0} \tag{3-86b}$$

$$\boxed{\frac{d^2h}{d\phi^2} = -m^2h} \tag{3-86c}$$

where m and n are constants (usually integers).

Solutions to (3-86a) through (3-86c) take the forms, respectively, of

$$f_1(r) = A_1 j_n(\beta r) + B_1 y_n(\beta r) \tag{3-87a}$$

or

$$f_2(r) = C_1 h_n^{(1)}(\beta r) + D_1 h_n^{(2)}(\beta r) \tag{3-87b}$$

and

$$g_1(\theta) = A_2 P_n^m(\cos\theta) + B_2 P_n^m(-\cos\theta) \qquad n \neq \text{integer} \tag{3-88a}$$

or

$$g_2(\theta) = C_2 P_n^m(\cos\theta) + D_2 Q_n^m(\cos\theta) \qquad n = \text{integer} \tag{3-88b}$$

and

$$h_1(\phi) = A_3 e^{-jm\phi} + B_3 e^{+jm\phi} \tag{3-89a}$$

or

$$h_2(\phi) = C_3\cos(m\phi) + D_3\sin(m\phi) \tag{3-89b}$$

In (3-87a) $j_n(\beta r)$ and $y_n(\beta r)$ are referred to, respectively, as the *spherical Bessel functions* of the first and second kind. They are used to represent radial standing waves, and they are related, respectively, to the corresponding regular Bessel functions $J_{n+1/2}(\beta r)$ and $Y_{n+1/2}(\beta r)$ by

$$j_n(\beta r) = \sqrt{\frac{\pi}{2\beta r}}J_{n+1/2}(\beta r) \tag{3-90a}$$

$$y_n(\beta r) = \sqrt{\frac{\pi}{2\beta r}}Y_{n+1/2}(\beta r) \tag{3-90b}$$

In (3-87b) $h_n^{(1)}(\beta r)$ and $h_n^{(2)}(\beta r)$ are referred to, respectively, as the *spherical Hankel functions* of the first and second kind. They are used to represent radial traveling waves, and they are related, respectively, to the regular Hankel functions $H_{n+1/2}^{(1)}(\beta r)$ and $H_{n+1/2}^{(2)}(\beta r)$ by

$$h_n^{(1)}(\beta r) = \sqrt{\frac{\pi}{2\beta r}}H_{n+1/2}^{(1)}(\beta r) \tag{3-91a}$$

$$h_n^{(2)}(\beta r) = \sqrt{\frac{\pi}{2\beta r}}H_{n+1/2}^{(2)}(\beta r) \tag{3-91b}$$

TABLE 3-3 Wave functions, zeroes, and infinities for radial waves in spherical coordinates

Wave type	Wave functions	Zeroes of wave functions	Infinities of wave functions
Traveling waves	$h_n^{(1)}(\beta r) = j_n(\beta r) + jy_n(\beta r)$ for $-r$ travel	$\beta r \to +j\infty$	$\beta r = 0$ $\beta r \to -j\infty$
	$h_n^{(2)}(\beta r) = j_n(\beta r) - jy_n(\beta r)$ for $+r$ travel	$\beta r \to -j\infty$	$\beta r = 0$ $\beta r \to +j\infty$
Standing waves	$j_n(\beta r)$ for $\pm r$	Infinite number	$\beta r \to \pm j\infty$
	$y_n(\beta r)$ for $\pm r$	Infinite number	$\beta r = 0$ $\beta r \to \pm j\infty$

Wave functions used to represent radial traveling and standing waves in spherical coordinates are listed in Table 3-3. More details on the spherical Bessel and Hankel functions can be found in Chapters 10 and 11 and Appendix IV.

In (3-88a) and (3-88b) $P_n^m(\cos\theta)$ and $Q_n^m(\cos\theta)$ are referred to, respectively, as the *associated Legendre functions* of the first and second kind (more details can be found in Chapter 10 and Appendix V).

The appropriate solution forms of f, g, and h will depend on the problem in question. For example, a typical solution for $\psi(r,\theta,\phi)$ of (3-85) to represent the fields within a sphere as shown in Figure 3-7 may take the form

$$\psi_1(r,\theta,\phi) = [A_1 j_n(\beta r) + B_1 y_n(\beta r)] \times [C_2 P_n^m(\cos\theta) + D_2 Q_n^m(\cos\theta)][C_3\cos(m\phi) + D_3\sin(m\phi)] \tag{3-92}$$

For the fields to be finite at $r = 0$, where $y_n(\beta r)$ possesses a singularity, and for any value of θ, including $\theta = 0, \pi$ where $Q_n^m(\cos\theta)$ possesses singularities, (3-92) reduces to

$$\psi_1(r,\theta,\phi) = A_{mn} j_n(\beta r) P_n^m(\cos\theta)[C_3\cos(m\phi) + D_3\sin(m\phi)] \tag{3-92a}$$

To represent the fields outside a sphere, like for scattering, a typical solution for $\psi(r,\theta,\phi)$ would take the form of

$$\psi_2(r,\theta,\phi) = B_{mn} h_n^{(2)}(\beta r) P_n^m(\cos\theta)[C_3\cos(m\phi) + D_3\sin(m\phi)] \tag{3-92b}$$

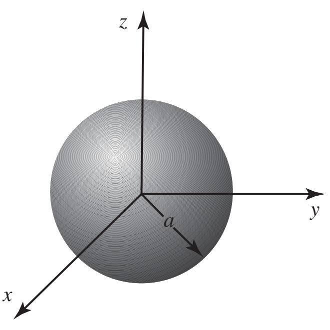

Figure 3-7 Geometry of a sphere of radius a.

whereby the spherical Hankel function of the second kind $h_n^{(2)}(\beta r)$ has replaced the spherical Bessel function of the first kind $j_n(\beta r)$ because outward traveling waves are formed outside the sphere, in contrast to the standing waves inside the sphere.

Other spherical Bessel and Hankel functions that are most often encountered in boundary-value electromagnetic problems are those utilized by Schelkunoff [3, 11]. These spherical Bessel and Hankel functions, denoted in general by $\hat{B}_n(\beta r)$ to represent any of them, must satisfy the differential equation

$$\frac{d^2\hat{B}_n}{dr^2} + \left[\beta^2 - \frac{n(n+1)}{r^2}\right]\hat{B}_n = 0 \tag{3-93}$$

The spherical Bessel and Hankel functions that are solutions to this equation are related to other spherical Bessel and Hankel functions of (3-90a) through (3-91b), denoted here by $b_n(\beta r)$, and to the regular Bessel and Hankel functions, denoted here by $B_{n+1/2}(\beta r)$, by

$$\hat{B}_n(\beta r) = \beta r\ b_n(\beta r) = \beta r\sqrt{\frac{\pi}{2\beta r}}B_{n+1/2}(\beta r) = \sqrt{\frac{\pi\beta r}{2}}B_{n+1/2}(\beta r) \tag{3-94}$$

More details concerning the application and properties of the spherical Bessel and Hankel functions can be found in Chapter 10.

3.5 MULTIMEDIA

On the website that accompanies this book, the following multimedia resources are included for the review, understanding and presentation of the material of this chapter.

- **Power Point (PPT)** viewgraphs, in multicolor.

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PROBLEMS

3.1. Derive the vector wave equations 3-16a and 3-16b for time-harmonic fields using the Maxwell equations of Table 1-4 for time-harmonic fields.

3.2. Verify that (3-28a) and (3-28b) are solutions to (3-26a).

3.3. Show that the second complex exponential in (3-31) represents a wave traveling in the $-z$ direction. Determine its phase velocity.

3.4. Using the method of separation of variables show that a solution to (3-37a) of the form (3-39) can be represented by (3-40a) through (3-43).

3.5. Show that the vector wave equation of (3-53) reduces, when **E** has a solution of the form (3-47), to the three scalar wave equations 3-54a through 3-54c.

3.6. Reduce (3-51) to (3-54a) through (3-54c) by expanding $\nabla^2\mathbf{E}$. Do not use (3-52); rather use the scalar Laplacian in cylindrical coordinates and treat **E** as a vector given by (3-47). Use that

$$\frac{\partial \hat{\mathbf{a}}_\rho}{\partial \rho} = \frac{\partial \hat{\mathbf{a}}_\phi}{\partial \rho} = \frac{\partial \hat{\mathbf{a}}_z}{\partial \rho} = 0 = \frac{\partial \hat{\mathbf{a}}_z}{\partial \phi} = \frac{\partial \hat{\mathbf{a}}_\rho}{\partial z} = \frac{\partial \hat{\mathbf{a}}_\phi}{\partial z} = \frac{\partial \hat{\mathbf{a}}_z}{\partial z}$$

$$\frac{\partial \hat{\mathbf{a}}_\rho}{\partial \phi} = \hat{\mathbf{a}}_\phi \qquad \frac{\partial \hat{\mathbf{a}}_\phi}{\partial \phi} = -\hat{\mathbf{a}}_\rho$$

3.7. Using large argument asymptotic forms, show that Bessel and Hankel functions represent, respectively, standing and traveling waves in the radial direction.

3.8. Using large argument asymptotic forms and assuming $e^{j\omega t}$ time convention, show that Hankel functions of the first kind represent traveling waves in the $-\rho$ direction whereas Hankel functions of the second kind represent traveling waves in the $+\rho$ direction. The opposite would be true were the time variations of the $e^{-j\omega t}$ form.

3.9. Using large argument asymptotic forms, show that Bessel functions of complex argument represent attenuating standing waves.

3.10. Assuming time variations of $e^{j\omega t}$ and using large argument asymptotic forms, show that Hankel functions of the first and second kind with complex arguments represent, respectively, attenuating traveling waves in the $-\rho$ and $+\rho$ directions.

3.11. Show that when **E** can be expressed as (3-71), the vector wave equation 3-53 reduces to the three scalar wave equations 3-74a through 3-74c.

3.12. Reduce (3-51) to (3-74a) through (3-74c) by expanding $\nabla^2\mathbf{E}$. Do not use (3-52); rather use the scalar Laplacian in spherical coordinates and treat **E** as a vector given by (3-71). Use that

$$\frac{\partial \hat{\mathbf{a}}_r}{\partial r} = \frac{\partial \hat{\mathbf{a}}_\theta}{\partial r} = \frac{\partial \hat{\mathbf{a}}_\phi}{\partial r} = 0$$

$$\frac{\partial \hat{\mathbf{a}}_r}{\partial \theta} = \hat{\mathbf{a}}_\theta \qquad \frac{\partial \hat{\mathbf{a}}_\theta}{\partial \theta} = -\hat{\mathbf{a}}_r \qquad \frac{\partial \hat{\mathbf{a}}_\phi}{\partial \theta} = 0$$

$$\frac{\partial \hat{\mathbf{a}}_r}{\partial \phi} = \sin\theta\,\hat{\mathbf{a}}_\phi \quad \frac{\partial \hat{\mathbf{a}}_\theta}{\partial \phi} = \cos\theta\,\hat{\mathbf{a}}_\phi$$

$$\frac{\partial \hat{\mathbf{a}}_\phi}{\partial \phi} = -\sin\theta\,\hat{\mathbf{a}}_r - \cos\theta\,\hat{\mathbf{a}}_\theta$$

3.13. Using large argument asymptotic forms, show that spherical Bessel functions represent standing waves in the radial direction.

3.14. Show that spherical Hankel functions of the first and second kind represent, respectively, radial traveling waves in the $-r$ and $+r$ directions. Assume time variations of $e^{j\omega t}$ and large argument asymptotic expansions for the spherical Hankel functions.

3.15. Justify that associated Legendre functions represent standing waves in the θ direction of the spherical coordinate system.

3.16. Verify the relation (3-94) between the various forms of the spherical Bessel and Hankel functions and the regular Bessel and Hankel functions.

CHAPTER 4

Wave Propagation and Polarization

4.1 INTRODUCTION

In Chapter 3 we developed the vector wave equations for the electric and magnetic fields in lossless and lossy media. Solutions to the wave equations were also demonstrated in rectangular, cylindrical, and spherical coordinates using the method of *separation of variables*. In this chapter we want to consider solutions for the electric and magnetic fields of time-harmonic waves that travel in infinite lossless and lossy media. In particular, we want to develop expressions for *transverse electromagnetic* (TEM) waves (or modes) traveling along principal axes and oblique angles. The parameters of wave impedance, phase and group velocities, and power and energy densities will be discussed for each.

The concept of wave polarization will be introduced, and the necessary and sufficient conditions to achieve linear, circular, and elliptical polarizations will be discussed and illustrated. The sense of rotation, clockwise (right-hand) or counterclockwise (left-hand), will also be introduced.

4.2 TRANSVERSE ELECTROMAGNETIC MODES

A *mode* is a particular field configuration. For a given electromagnetic boundary-value problem, many field configurations that satisfy the wave equations, Maxwell's equations, and the boundary conditions usually exist. All these different field configurations (solutions) are usually referred to as *modes*.

A TEM mode is one whose field intensities, both **E** (electric) and **H** (magnetic), at every point in space are contained on a local plane, referred to as *equiphase plane*, that is independent of time. In general, the orientations of the local planes associated with the TEM wave are different at different points in space. In other words, at point (x_1, y_1, z_1) all the field components are contained on a plane. At another point (x_2, y_2, z_2) all field components are again contained on a plane; however, the two planes need not be parallel. This is illustrated in Figure 4-1a.

If the space orientation of the planes for a TEM mode is the same (equiphase planes are parallel), as shown in Figure 4-1b, then the fields form *plane waves*. In other words, the equiphase surfaces are parallel planar surfaces. If in addition to having planar equiphases the field has equiamplitude planar surfaces (the amplitude is the same over each plane), then it is called a *uniform plane wave*; that is, the field is not a function of the coordinates that form the equiphase and equiamplitude planes.

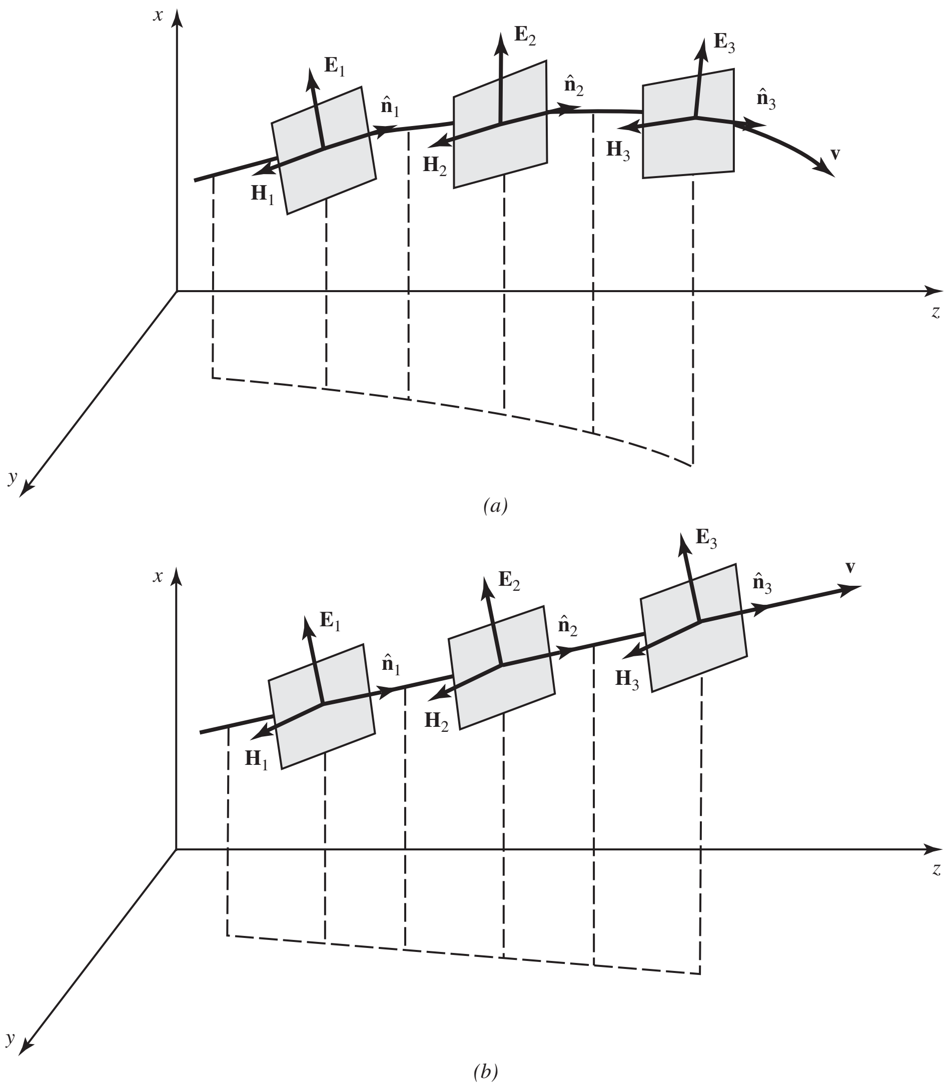
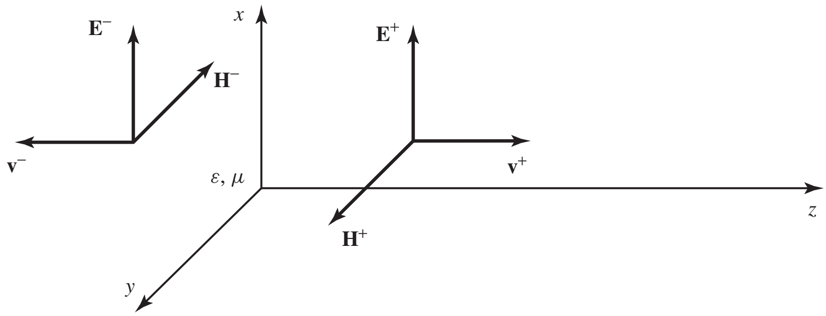


Figure 4-1 Phase fronts of waves. (*a*) TEM. (*b*) Plane.

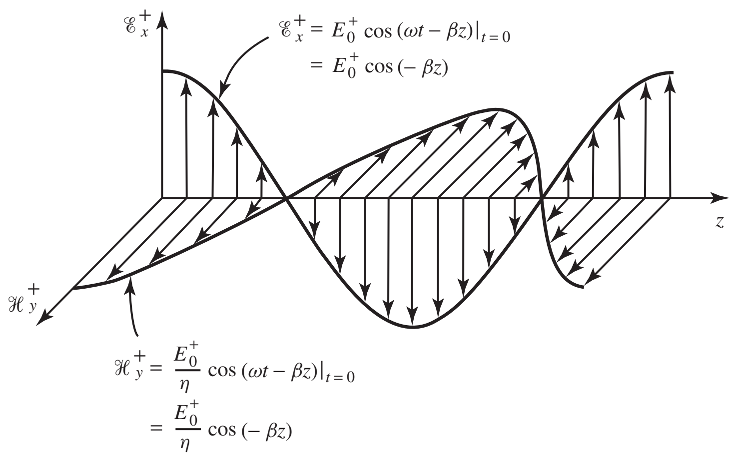
4.2.1 Uniform Plane Waves in an Unbounded Lossless Medium — Principal Axis

In this section we will write expressions for the electric and magnetic fields of a uniform plane wave traveling in an unbounded medium. In addition the wave impedance, phase and energy (group) velocities, and power and energy densities of the wave will be discussed.

A. Electric and Magnetic Fields Let us assume that a time-harmonic uniform plane wave is traveling in an unbounded lossless medium (ε, μ) in the z direction (either positive or negative), as shown in Figure 4-2a. In addition, for simplicity, let us assume the electric field of the wave has only an x component. We want to write expressions for the electric and magnetic fields associated with this wave.



(a)



(b)

Figure 4-2 Uniform plane wave fields. (*a*) Complex. (*b*) Instantaneous.

For the electric and magnetic field components to be valid solutions of a time-harmonic electromagnetic wave, they must satisfy Maxwell's equations as given in Table 1-4 or the corresponding wave equations as given, respectively, by (3-18a) and (3-18b). Here the approach will be to initiate the solution by solving the wave equation for either the electric or magnetic field and then finding the other field using Maxwell's equations. An alternate procedure, which has been assigned as an end-of-chapter problem, would be to follow the entire solution using only Maxwell's equations.

Since the electric field has only an x component, it must satisfy the scalar wave equation of (3-20a) or (3-22), whose general solution is given by (3-23). Because the wave is a uniform plane wave that travels in the z direction, its solution is not a function of x and y. Therefore (3-23) reduces to

$$E_x(z) = h(z) \tag{4-1}$$

The solutions of $h(z)$ are given by (3-30a) or (3-30b). Since the wave in question is a traveling wave, instead of a standing wave, its most appropriate solution is that given by (3-30a). The first term in (3-30a) represents a wave that travels in the $+z$ direction and the second term represents a wave that travels in the $-z$ direction. Therefore the solution of (4-1), using (3-30a), can be written as

$$E_x(z) = A_3 e^{-j\beta z} + B_3 e^{+j\beta z} = E_x^+ + E_x^- \tag{4-2}$$

or

$$E_x(z) = E_0^+ e^{-j\beta z} + E_0^- e^{+j\beta z} = E_x^+ + E_x^- \tag{4-2a}$$

$$E_x^+(z) = E_0^+ e^{-j\beta z} \tag{4-2b}$$

$$E_x^-(z) = E_0^- e^{+j\beta z} \tag{4-2c}$$

since $\beta_z = \beta$ because $\beta_x = \beta_y = 0$. E_0^+ and E_0^- represent, respectively, the amplitudes of the positive and negative (in the z direction) traveling waves.

The corresponding magnetic field must also be a solution of its wave equation 3-18b, and its form will be similar to (4-2). However, since we do not know which components of magnetic field coexist with the x component of the electric field, they are most appropriately determined by using one of Maxwell's equations as given in Table 1-4. Since the electric field is known, as given by (4-2), the magnetic field can best be found using

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{4-3}$$

or

$$\mathbf{H} = -\frac{1}{j\omega\mu}\nabla \times \mathbf{E} = -\frac{1}{j\omega\mu}\begin{bmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & 0 & 0 \end{bmatrix} \tag{4-3a}$$

which, using (4-2a), reduces to

$$\begin{aligned} \mathbf{H} &= -\hat{\mathbf{a}}_y \frac{1}{j\omega\mu}\left\{\frac{\partial E_x}{\partial z}\right\} = \hat{\mathbf{a}}_y \frac{\beta}{\omega\mu}\left\{E_0^+ e^{-j\beta z} - E_0^- e^{+j\beta z}\right\} \\ \mathbf{H} &= \hat{\mathbf{a}}_y \frac{1}{\sqrt{\mu/\varepsilon}}\left\{E_0^+ e^{-j\beta z} - E_0^- e^{+j\beta z}\right\} = \hat{\mathbf{a}}_y \frac{1}{\sqrt{\mu/\varepsilon}}\left\{E_x^+ - E_x^-\right\} = \hat{\mathbf{a}}_y \left\{H_y^+ + H_y^-\right\} \end{aligned} \tag{4-3b}$$

where

$$H_y^+ = \frac{1}{\sqrt{\mu/\varepsilon}}E_x^+ \tag{4-3c}$$

$$H_y^- = -\frac{1}{\sqrt{\mu/\varepsilon}}E_x^- \tag{4-3d}$$

Plots of the instantaneous *positive* traveling electric and magnetic fields at $t = 0$ as a function of z are shown in Figure 4-2*b*. Similar plots can be drawn for the negative traveling fields.

B. Wave Impedance Since each term for the magnetic field (A/m) in (4-3c) and (4-3d) is individually identical to the corresponding term for the electric field (V/m) in (4-2a), the factor $\sqrt{\mu/\varepsilon}$ in the denominator in (4-3c) and (4-3d) must have units of ohms (V/A). Therefore the factor $\sqrt{\mu/\varepsilon}$ is known as the *wave impedance*, Z_w, denoted by the ratio of the electric to magnetic field, and it is usually represented by η

$$Z_w = \frac{E_x^+}{H_y^+} = -\frac{E_x^-}{H_y^-} = \eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{4-4}$$

The wave impedance of (4-4) is identical to a quantity that is referred to as the *intrinsic impedance* $\eta = \sqrt{\mu/\varepsilon}$ of the medium. In general, this is true not only for uniform plane waves but also for plane and TEM waves; however, it is not true for TE or TM modes.

In (4-3d) it is also observed that a negative sign is found in front of the magnetic field component that travels in the $-z$ direction; a positive sign is noted in front of the positive traveling wave. The general procedure that can be followed to find the magnetic field components, given the electric field components, or to find the electric field components, given the magnetic field components, is the following:

1. Place the fingers of your right hand in the direction of the electric field component.
2. Direct your thumb toward the direction of wave travel (power flow).
3. Rotate your fingers 90° in a direction so that a right-hand screw is formed.
4. The new direction of your fingers is the direction of the magnetic field component.
5. Divide the electric field component by the wave impedance to obtain the corresponding magnetic field component.

The foregoing procedure must be followed for each term of each component of an electric or magnetic field. The results are identical to those that would be obtained by using Maxwell's equations. If the wave impedance is known in advance, as it is for TEM waves, this procedure is simpler and much more rapid than using Maxwell's equations. By following this procedure, the answers (including the signs) in (4-3c) and (4-3d) given (4-2b) and (4-2c) are obvious.

To illustrate the procedure, let us consider another example.

Example 4-1

The electric field of a uniform plane wave traveling in free space is given by

$$\mathbf{E} = \hat{\mathbf{a}}_y \left(E_0^+ e^{-j\beta z} + E_0^- e^{+j\beta z}\right) = \hat{\mathbf{a}}_y \left(E_y^+ + E_y^-\right)$$

where E_0^+ and E_0^- are constants. Find the corresponding magnetic field using the outlined procedure.

Solution: For the electric field component that is traveling in the $+z$ direction, the corresponding magnetic field component is given by

$$\mathbf{H}^+ = -\hat{\mathbf{a}}_x \frac{E_0^+}{\eta_0} e^{-j\beta z} \simeq -\hat{\mathbf{a}}_x \frac{E_0^+}{377} e^{-j\beta z}$$

where

$$\eta_0 = Z_w = \sqrt{\frac{\mu_0}{\varepsilon_0}} \simeq 377\,\text{ohms}$$

Similarly, for the wave that is traveling in the $-z$ direction we can write that

$$\mathbf{H}^- = \hat{\mathbf{a}}_x \frac{E_0^-}{\eta_0} e^{+j\beta z} \simeq \hat{\mathbf{a}}_x \frac{E_0^-}{377} e^{+j\beta z}$$

Therefore the total magnetic field is equal to

$$\mathbf{H} = \mathbf{H}^+ + \mathbf{H}^- = \hat{\mathbf{a}}_x \frac{1}{\eta_0} \left(-E_0^+ e^{-j\beta z} + E_0^- e^{+j\beta z}\right)$$

The same answer would be obtained if Maxwell's equations were used, and it is assigned as an end-of-chapter problem.

The term in the expression for the electric field in (4-2a) that identifies the direction of wave travel can also be written in vector notation. This is usually more convenient to use when dealing with waves traveling at oblique angles. Equation 4-2a can therefore take the more general form of

$$E_x(z) = E_0^+ e^{-j\boldsymbol{\beta}^+ \cdot \mathbf{r}} + E_0^- e^{-j\boldsymbol{\beta}^- \cdot \mathbf{r}} \tag{4-5}$$

where

$$\boldsymbol{\beta}^+ = \hat{\boldsymbol{\beta}}^+ \beta = \hat{\mathbf{a}}_x \beta_x^+ + \hat{\mathbf{a}}_y \beta_y^+ + \hat{\mathbf{a}}_z \beta_z^+ \Big|_{\substack{\beta_x^+ = \beta_y^+ = 0 \\ \beta_z^+ = \beta}} = \hat{\mathbf{a}}_z \beta \tag{4-5a}$$

$$\boldsymbol{\beta}^- = \hat{\boldsymbol{\beta}}^- \beta = \hat{\mathbf{a}}_x \beta_x^- + \hat{\mathbf{a}}_y \beta_y^- - \hat{\mathbf{a}}_z \beta_z^- \Big|_{\substack{\beta_x^- = \beta_y^- = 0 \\ \beta_z^- = \beta}} = -\hat{\mathbf{a}}_z \beta \tag{4-5b}$$

$$\mathbf{r} = \text{position vector} = \hat{\mathbf{a}}_x x + \hat{\mathbf{a}}_y y + \hat{\mathbf{a}}_z z \tag{4-5c}$$

In (4-5a) through (4-5c), β_x, β_y, β_z represent, respectively, the phase constants of the wave in the x, y, z directions, $\mathbf{r}$ represents the position vector in rectangular coordinates, and $\hat{\boldsymbol{\beta}}^+$ and $\hat{\boldsymbol{\beta}}^-$ represent unit vectors in the directions of $\boldsymbol{\beta}^+$ and $\boldsymbol{\beta}^-$. The notation used in (4-5) through (4-5c) to represent the wave travel will be most convenient to express wave travel at oblique angles, as will be the case in Section 4.2.2.

C. Phase and Energy (Group) Velocities, Power, and Energy Densities The expressions for the electric and magnetic fields, as given by (4-2a) and (4-3b), represent the spatial variations of the field intensities. The corresponding instantaneous forms of each can be written, using (1-61a) and (1-61b) and assuming E_0^+ and E_0^- are real constants, as

$$\begin{aligned} \mathscr{E}_x(z;t) &= \mathscr{E}_x^+(z;t) + \mathscr{E}_x^-(z;t) = \text{Re}\left[E_0^+ e^{-j\beta z} e^{j\omega t}\right] + \text{Re}\left[E_0^- e^{+j\beta z} e^{j\omega t}\right] \\ &= E_0^+ \cos(\omega t - \beta z) + E_0^- \cos(\omega t + \beta z) \end{aligned} \tag{4-6a}$$

$$\begin{aligned} \mathscr{H}_y(z;t) &= \mathscr{H}_y^+(z;t) + \mathscr{H}_y^-(z;t) \\ &= \frac{1}{\sqrt{\mu/\varepsilon}}\left[E_0^+ \cos(\omega t - \beta z) - E_0^- \cos(\omega t + \beta z)\right] \end{aligned} \tag{4-6b}$$

In each of the fields, as given by (4-6a) and (4-6b), the first term represents, according to (3-34) through (3-35) and Figure 3-3, a wave that travels in the $+z$ direction; the second term represents a wave that travels in the $-z$ direction. To maintain a constant phase in the first term of (4-6a), the velocity must be equal, according to (3-35), to

$$v_p^+ = +\frac{dz}{dt} = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{4-7}$$

The corresponding velocity of the second term in (4-6a) is identical in magnitude to (4-7) but with a negative sign to reflect the direction of wave travel. The velocity of (4-7) is referred to as the *phase velocity*, and it represents the velocity that must be maintained in order to keep in step with a constant phase front of the wave. As will be shown for oblique traveling waves, the phase velocity of such waves can exceed the velocity of light. This is only a hypothetical speed, as will be explained in Section 4.2.2C. Aside of nonuniform plane waves, also referred to as slow surface waves (see Section 5.3.4A), in general the phase velocity can be equal to or even greater than the speed of light. Variations of the instantaneous positive traveling electric $\mathscr{E}_x^+(z;t)$ and magnetic $\mathscr{H}_y^+(z;t)$ fields as a function of z for $t=0$ are shown in Figure 4-2*b*. As time increases, both curves will shift in the positive z direction. A similar set of curves can be drawn for the negative traveling electric $\mathscr{E}_x^-(z;t)$ and magnetic $\mathscr{H}_y^-(z;t)$ fields.

The electric and magnetic energies (W-s/m^3) and power densities (W/m^2) associated with the positive traveling waves of (4-6a) and (4-6b) can be written, according to (l-58f) and (l-58e), as

$$w_e^+ = \frac{1}{2}\varepsilon\mathscr{E}_x^{+2} = \frac{1}{2}\varepsilon E_0^{+2}\cos^2(\omega t - \beta z) \tag{4-8a}$$

$$w_m^+ = \frac{1}{2}\mu\mathscr{H}_y^{+2} = \frac{1}{2}\mu\left[(\varepsilon/\mu)E_0^{+2}\cos^2(\omega t - \beta z)\right] = \frac{1}{2}\varepsilon E_0^{+2}\cos^2(\omega t - \beta z) \tag{4-8b}$$

$$\begin{aligned}\boldsymbol{\mathscr{S}}^+ &= \boldsymbol{\mathscr{E}}^+ \times \boldsymbol{\mathscr{H}}^+ = \hat{\mathbf{a}}_x E_0^+ \cos(\omega t - \beta z) \times \left[\hat{\mathbf{a}}_y\left(1/\sqrt{\mu/\varepsilon}\right)E_0^+\cos(\omega t - \beta z)\right] \\ &= \hat{\mathbf{a}}_z \mathscr{S}^+ = \hat{\mathbf{a}}_z\left(1/\sqrt{\mu/\varepsilon}\right)E_0^{+2}\cos^2(\omega t - \beta z)\end{aligned} \tag{4-8c}$$

The ratio formed by dividing the power density $\mathscr{S}$ (W/m^2) by the total energy density $w = w_e + w_m$ (J/m^3 = W-s/m^3) is referred to as the *energy* (*group*) *velocity* v_e, and it is given by

$$v_e^+ = \frac{\mathscr{S}^+}{w^+} = \frac{\mathscr{S}^+}{w_e^+ + w_m^+} = \frac{\left(1/\sqrt{\mu/\varepsilon}\right)E_0^{+2}\cos^2(\omega t - \beta z)}{\varepsilon E_0^{+2}\cos^2(\omega t - \beta z)} = \frac{1}{\sqrt{\mu\varepsilon}} \tag{4-9}$$

The energy velocity represents the velocity with which the wave energy is transported. It is apparent that (4-9) is identical to (4-7). In general that is not the case. In fact, the energy velocity v_e^+ can be equal to, but not exceed, the speed of light, and the product of the phase velocity v_p and energy velocity v_e must always be equal to

$$\boxed{v_p^+ v_e^+ = (v^+)^2 = \frac{1}{\mu\varepsilon}} \tag{4-10}$$

where $v^+ = 1/\sqrt{\mu\varepsilon}$ is the speed of light. The same holds for the negative traveling waves.

The time-average power density (Poynting vector) associated with the positive traveling wave can be written, using (l-70) and the first terms of (4-2a) and (4-3b), as

$$\boldsymbol{\mathscr{S}}_{\text{av}}^+ = \frac{1}{2}\,\text{Re}\left(\mathbf{E}^+ \times \mathbf{H}^{+*}\right) = \hat{\mathbf{a}}_z \frac{1}{2\sqrt{\mu/\varepsilon}}|E_x^+|^2 = \hat{\mathbf{a}}_z\frac{|E_0^+|^2}{2\sqrt{\mu/\varepsilon}} = \hat{\mathbf{a}}_z\frac{|E_0^+|^2}{2\eta} \tag{4-11}$$

A similar expression is derived for the negative traveling wave.

D. Standing Waves Each of the terms in (4-2a) and (4-3b) represents individually *traveling* waves, the first traveling in the positive z direction and the second in the negative z direction. The two together form a so-called *standing wave*, which is comprised of two oppositely traveling waves.

To examine the characteristics of a standing wave, let us rewrite (4-2a) as

$$\begin{aligned}E_x(z) &= E_0^+ e^{-j\beta z} + E_0^- e^{+j\beta z} \\ &= E_0^+\left[\cos(\beta z) - j\sin(\beta z)\right] + E_0^-\left[\cos(\beta z) + j\sin(\beta z)\right] \\ &= \left(E_0^+ + E_0^-\right)\cos(\beta z) - j\left(E_0^+ - E_0^-\right)\sin(\beta z)\end{aligned}$$

$$E_x(z) = \sqrt{\left(E_0^+ + E_0^-\right)^2 \cos^2(\beta z) + \left(E_0^+ - E_0^-\right)^2 \sin^2(\beta z)}$$
$$\times \exp\left\{-j \tan^{-1}\left[\frac{\left(E_0^+ - E_0^-\right)\sin(\beta z)}{\left(E_0^+ + E_0^-\right)\cos(\beta z)}\right]\right\}$$
$$E_x(z) = \sqrt{\left(E_0^+\right)^2 + \left(E_0^-\right)^2 + 2E_0^+E_0^- \cos(2\beta z)}$$
$$\times \exp\left\{-j \tan^{-1}\left[\frac{\left(E_0^+ - E_0^-\right)}{\left(E_0^+ + E_0^-\right)}\tan(\beta z)\right]\right\} \tag{4-12}$$

The amplitude of the waveform given by (4-12) is equal to

$$|E_x(z)| = \sqrt{\left(E_0^+\right)^2 + \left(E_0^-\right)^2 + 2E_0^+E_0^- \cos(2\beta z)} \tag{4-12a}$$

By examining (4-12a), it is evident that its maximum and minimum values are given, respectively, by

$$|E_x(z)|_{\max} = |E_0^+| + |E_0^-| \text{ when } \beta z = m\pi, m = 0, 1, 2, \ldots \tag{4-13a}$$

and for $|E_0^+| > |E_0^-|$,

$$|E_x(z)|_{\min} = |E_0^+| - |E_0^-| \text{ when } \beta z = \frac{(2m+1)\pi}{2}, m = 0, 1, 2, \ldots \tag{4-13b}$$

Neighboring maximum and minimum values are separated by a distance of $\lambda/4$ or successive maxima or minima are separated by $\lambda/2$.

The instantaneous field of (4-12) can also be written as

$$\mathscr{E}_x(z;t) = \text{Re}\left[E_x(z)\,e^{j\omega t}\right]$$
$$= \sqrt{\left(E_0^+\right)^2 + \left(E_0^-\right)^2 + 2E_0^+E_0^- \cos(2\beta z)}$$
$$\times \cos\left[\omega t - \tan^{-1}\left\{\frac{E_0^+ - E_0^-}{E_0^+ + E_0^-}\tan(\beta z)\right\}\right] \tag{4-14}$$

It is apparent that (4-12a) represents the envelope of the maximum values the instantaneous field of (4-14) will achieve as a function of time at a given position. Since this envelope of maximum values does not move (change) in position as a function of time, it is referred to as the *standing wave pattern* and the associated wave of (4-12) or (4-14) is referred to as the *standing wave*.

The ratio of the maximum/minimum values of the standing wave pattern of (4-12a), as given by (4-13a) and (4-13b), is referred to as the standing wave ratio (SWR), and it is given by

$$\text{SWR} = \frac{|E_x(z)|_{\max}}{|E_x(z)|_{\min}} = \frac{|E_0^+| + |E_0^-|}{|E_0^+| - |E_0^-|} = \frac{1 + \frac{|E_0^-|}{|E_0^+|}}{1 - \frac{|E_0^-|}{|E_0^+|}} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \tag{4-15}$$

where Γ is the reflection coefficient. Since in transmission lines we usually deal with voltages and currents (instead of electric and magnetic fields), the SWR is usually referred to as the VSWR (voltage standing wave ratio). Plots of the standing wave pattern in terms of E_0^+ as a function of $z(-\lambda \le z \le \lambda)$ for $|\Gamma| = 0$, 0.2, 0.4, 0.6, 0.8, and 1 are shown in Figure 4-3.

The SWR is a quantity that can be measured with instrumentation [1, 2]. SWR has values in the range of $1 \le \text{SWR} \le \infty$. The value of the SWR indicates the amount of interference between the two opposite traveling waves; the smaller the SWR value, the lesser the interference.

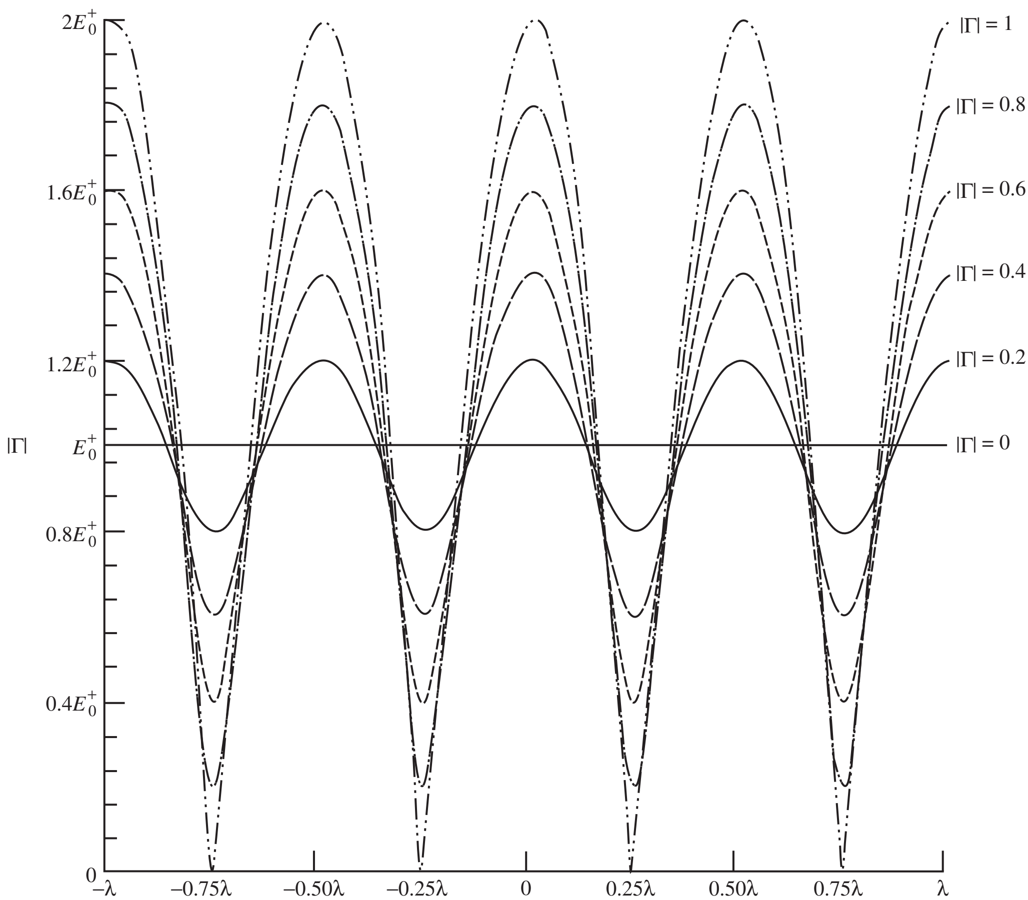


Figure 4-3 Standing wave pattern as a function of distance for a uniform plane wave with different reflection coefficients.

The minimum SWR value of unity occurs when $|\Gamma| = E_0^- / E_0^+ = 0$, and it indicates that no interference is formed. Thus the standing wave reduces to a pure traveling wave. The maximum SWR of infinity occurs when $|\Gamma| = E_0^- / E_0^+ = 1$, and it indicates that the negative traveling wave is of the same intensity as the positive traveling wave. This provides the maximum interference, and the wave forms a pure standing wave pattern given by

$$|E_x(z)|_{E_0^+ = E_0^-} = 2E_0^+ |\cos(\beta z)| = 2E_0^- |\cos(\beta z)| \tag{4-16}$$

The pattern of this is a rectified cosine function, and it is represented in Figure 4-3 by the $|\Gamma| = 1$ curve. The pattern exhibits pure nulls and peak values of twice the amplitude of the incident wave.

4.2.2 Uniform Plane Waves in an Unbounded Lossless Medium — Oblique Angle

In this section, expressions for the electric and magnetic fields, wave impedance, phase and group velocities, and power and energy densities will be written for uniform plane waves traveling at oblique angles in an unbounded medium. All of these will be done for waves that are uniform plane waves to the direction of travel.

A. Electric and Magnetic Fields Let us assume that a uniform plane wave is traveling in an unbounded medium in a direction shown in Figure 4-4*a*. The amplitudes of the positive and negative traveling electric fields are E_0^+ and E_0^-, respectively, and the assumed directions of each are also illustrated in Figure 4-4*a*. It is desirable to write expressions for the positive and negative traveling electric and magnetic field components.

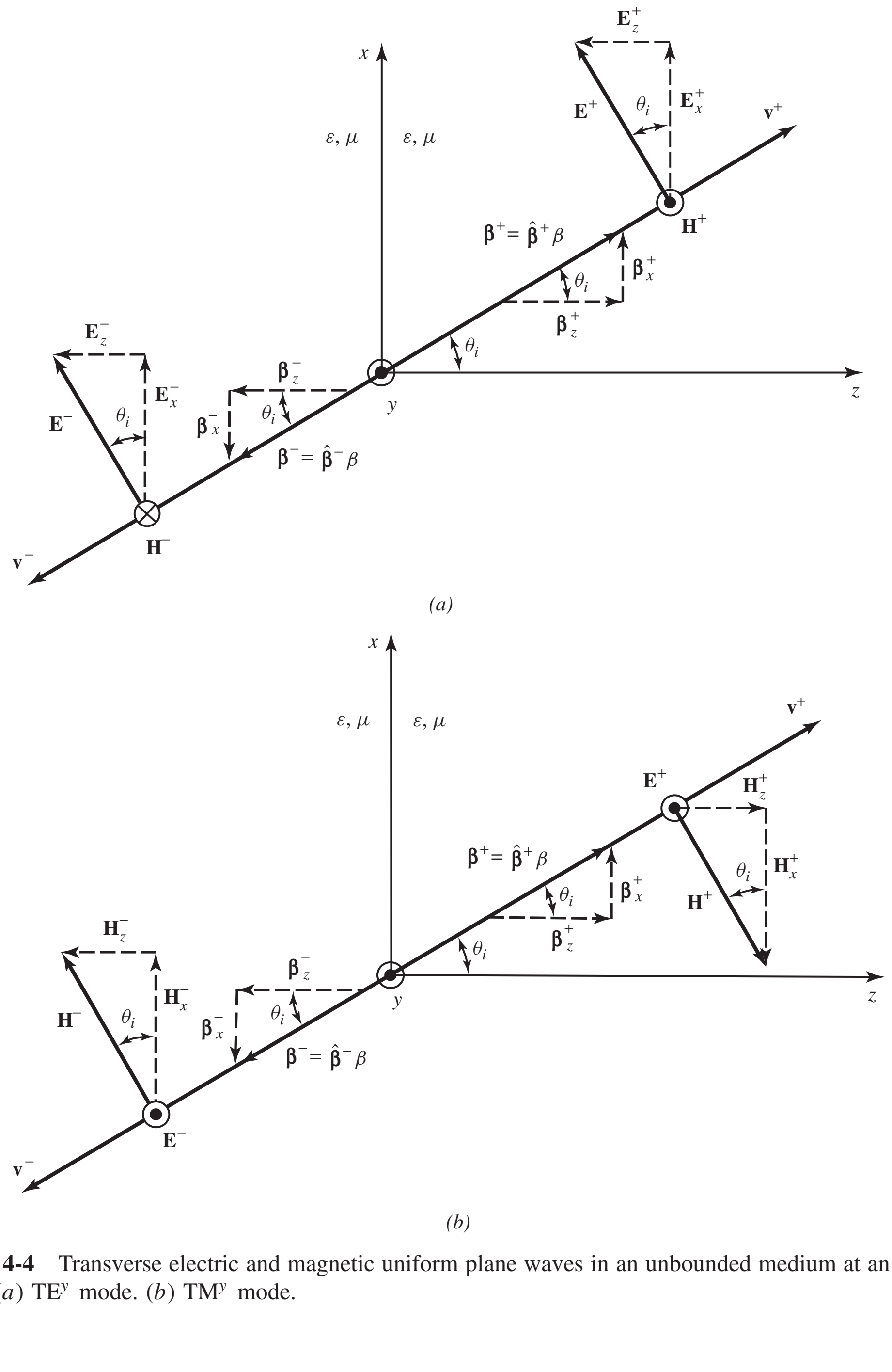


Figure 4-4 Transverse electric and magnetic uniform plane waves in an unbounded medium at an oblique angle. (*a*) TE^y mode. (*b*) TM^y mode.

Since the electric field of the wave of Figure 4-4*a* does not have a y component, the field configuration is referred to as *transverse electric to y* (TE^y). More detailed discussion on the construction of *transverse electric* (TE) and *transverse magnetic* (TM) field configurations, as well as *transverse electromagnetic* (TEM), can be found in Chapter 6.

Because for the TE^y wave of Figure 4-4*a* the electric field is pointing along a direction that does not coincide with any of the principal axes, it can be decomposed into components coincident with the principal axes. According to the geometry of Figure 4-4*a*, it is evident that the electric field can be written as

$$\begin{aligned}\mathbf{E} = \mathbf{E}^+ + \mathbf{E}^- &= E_0^+ (\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\, e^{-j\boldsymbol{\beta}^+ \cdot \mathbf{r}} \\ &\quad + E_0^- (\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\, e^{-j\boldsymbol{\beta}^- \cdot \mathbf{r}} \end{aligned} \tag{4-17}$$

where $\mathbf{r}$ is the position vector of (4-5c), and it is displayed graphically in Figure 4-5. Since the phase constants $\boldsymbol{\beta}^+$ and $\boldsymbol{\beta}^-$ can be written, respectively, as

$$\boldsymbol{\beta}^+ = \hat{\boldsymbol{\beta}}^+ \beta = \hat{\mathbf{a}}_x \beta_x^+ + \hat{\mathbf{a}}_z \beta_z^+ = \beta\,(\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i) \tag{4-17a}$$

$$\boldsymbol{\beta}^- = \hat{\boldsymbol{\beta}}^- \beta = \hat{\mathbf{a}}_x \beta_x^- + \hat{\mathbf{a}}_z \beta_z^- = -\beta\,(\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i) \tag{4-17b}$$

(4-17) can be expressed as

$$\begin{aligned}\mathbf{E} &= E_0^+ (\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\, e^{-j\beta(x\sin\theta_i + z\cos\theta_i)} \\ &\quad + E_0^- (\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\, e^{+j\beta(x\sin\theta_i + z\cos\theta_i)} \end{aligned} \tag{4-18a}$$

Since the wave is a uniform plane wave, the amplitude of its magnetic field is related to the amplitude of its electric field by the wave impedance (in this case also by the intrinsic

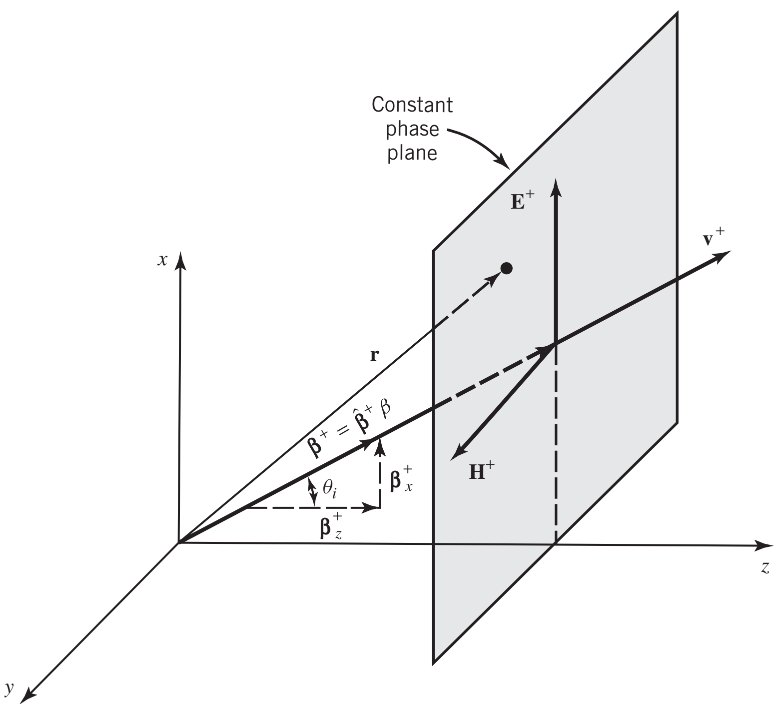


Figure 4-5 Phase front of a TEM wave traveling in a general direction.

impedance) as given by (4-4). Since the magnetic field is traveling in the same direction as the electric field, the exponentials used to indicate its directions of travel are the same as those of the electric field as given in (4-18a). The directions of the magnetic field can be found using the right-hand procedure outlined in Section 4.2.1 and illustrated graphically in Figure 4-2*b* for the positive traveling wave. Using all of the preceding information, it is evident that the magnetic field corresponding to the electric field of (4-18a) can be written as

$$\mathbf{H} = \mathbf{H}^+ + \mathbf{H}^- = \hat{\mathbf{a}}_y \left[\frac{E_0^+}{\eta} e^{-j\beta(x\sin\theta_i + z\cos\theta_i)} - \frac{E_0^-}{\eta} e^{+j\beta(x\sin\theta_i + z\cos\theta_i)} \right] \tag{4-18b}$$

In vector form, (4-18b) can also be written as

$$\mathbf{H} = \frac{1}{\eta}\left[\hat{\boldsymbol{\beta}}^+ \times \mathbf{E}^+ + \hat{\boldsymbol{\beta}}^- \times \mathbf{E}^-\right] \tag{4-18c}$$

The same form can be used to relate the **E** and **H** for any TEM wave traveling in any direction. It is apparent that when $\theta_i = 0$, (4-18a) and (4-18b) reduce to (4-2a) and (4-3b), respectively. The same answer for the magnetic field of (4-18b) can be obtained by applying Maxwell's equation 4-3 to the electric field of (4-18a). This is left for the reader as an end-of-the-chapter exercise.

The planes of constant phase at any time t are obtained by setting the phases of (4-18a) or (4-18b) equal to a constant, that is

$$\boldsymbol{\beta}^+ \cdot \mathbf{r} = \beta_x^+ x + \beta_y^+ y + \beta_z^+ z|_{y=0} = \beta\,(x\sin\theta_i + z\cos\theta_i) = C^+ \tag{4-19a}$$

$$\boldsymbol{\beta}^- \cdot \mathbf{r} = \beta_x^- x + \beta_y^- y + \beta_z^- z|_{y=0} = -\beta\,(x\sin\theta_i + z\cos\theta_i) = C^- \tag{4-19b}$$

Each of (4-19a) and (4-19b) are equations of a plane in either the spherical or rectangular coordinates with unit vectors $\hat{\boldsymbol{\beta}}^+$ and $\hat{\boldsymbol{\beta}}^-$ normal to each of the respective surfaces. The respective phase velocities in any direction (r, x, or z) are obtained by letting

$$\boldsymbol{\beta}^+ \cdot \mathbf{r} - \omega t = \beta\,(x\sin\theta_i + z\cos\theta_i) - \omega t = C_0^+ \tag{4-19c}$$

$$\boldsymbol{\beta}^- \cdot \mathbf{r} - \omega t = -\beta\,(x\sin\theta_i + z\cos\theta_i) - \omega t = C_0^- \tag{4-19d}$$

and taking a derivative with respect to time.

Example 4-2

Another exercise of interest is that in which the electric field is directed along the $+y$ direction and the wave is traveling along an oblique angle θ_i, as shown in Figure 4-4*b*. This is referred to as a TM^y wave. The objective here is again to write expressions for the positive and negative electric and magnetic field components, assuming the amplitudes of the positive and negative electric field components are E_0^+ and E_0^-, respectively.

Solution: Since this wave only has a y electric field component, and it is traveling in the same direction as that of Figure 4-4*a*, we can write the electric field as

$$\mathbf{E} = \mathbf{E}^+ + \mathbf{E}^- = \hat{\mathbf{a}}_y \left[E_0^+ e^{-j\beta(x\sin\theta_i + z\cos\theta_i)} + E_0^- e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}\right]$$

Using the right-hand procedure outlined in Section 4.2.1, the corresponding magnetic field components are pointed along directions indicated in Figure 4-4*b*. Since the magnetic field is not directed along any of the principal axes, it can be decomposed into components that coincide with the principal axes, as shown in Figure 4-4*b*. Doing this and relating the amplitude of the electric and magnetic fields by the intrinsic impedance, we can write the magnetic field as

$$\mathbf{H} = \mathbf{H}^+ + \mathbf{H}^- = \frac{E_0^+}{\eta}\,(-\hat{\mathbf{a}}_x \cos\theta_i + \hat{\mathbf{a}}_z \sin\theta_i)\, e^{-j\beta(x\sin\theta_i + z\cos\theta_i)}$$
$$+\frac{E_0^-}{\eta}\,(\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\, e^{+j\beta(x\sin\theta_i + z\cos\theta_i)}$$

The same answers could have been obtained if Maxwell's equation 4-3 were used. Since the magnetic field does not have any y components, this field configuration is referred to as *transverse magnetic to y* (TM^y), which will be discussed in more detail in Chapter 6.

B. Wave Impedance Since the TE^y and TM^y fields of Section 4.2.2A were TEM to the direction of travel, the wave impedance of each in the direction $\boldsymbol{\beta}$ of wave travel is the same as the intrinsic impedance of the medium. However, there are other directional impedances toward the x and z directions. These impedances are obtained by dividing the electric field component by the corresponding orthogonal magnetic field component. These two components are chosen so that the cross product of the electric field and the magnetic field, which corresponds to the direction of power flow, is in the direction of the wave travel.

Following the aforementioned procedure, the directional impedances for the TE^y fields of (4-18a) and (4-18b) can be written as

$$\underline{TE^y}$$

$$Z_x^+ = -\frac{E_z^+}{H_y^+} = \eta\sin\theta_i = Z_x^- = \frac{E_z^-}{H_y^-} \tag{4-20a}$$

$$Z_z^+ = \frac{E_x^+}{H_y^+} = \eta\cos\theta_i = Z_z^- = -\frac{E_x^-}{H_y^-} \tag{4-20b}$$

In the same manner, the directional impedances of the TM^y fields of Example 4-2 can be written as

$$\underline{TM^y}$$

$$Z_x^+ = \frac{E_y^+}{H_z^+} = \frac{\eta}{\sin\theta_i} = Z_x^- = -\frac{E_y^-}{H_z^-} \tag{4-21a}$$

$$Z_z^+ = -\frac{E_y^+}{H_x^+} = \frac{\eta}{\cos\theta_i} = Z_z^- = \frac{E_y^-}{H_x^-} \tag{4-21b}$$

It is apparent from the preceding results that the directional impedances of the TE^y oblique incidence traveling waves are equal to or smaller than the intrinsic impedance and those of the TM^y are equal to or larger than the intrinsic impedance. In addition, the positive and negative directional impedances of the same orientation are the same. This is the main principle of the *transverse resonance method* (see Section 8.6), which is used to analyze microwave circuits and antenna systems [3, 4].

C. Phase and Energy (Group) Velocities The wave velocity v_r of the fields given by (4-18a) and (4-18b) in the direction $\boldsymbol{\beta}$ of travel is equal to the speed of light v. Since the wave is a plane wave to the direction $\boldsymbol{\beta}$ of travel, the planes over which the phase is constant (constant phase planes) are perpendicular to the direction $\boldsymbol{\beta}$ of wave travel. This is illustrated graphically in Figure 4-6. To maintain a constant phase (or to keep in step with a constant phase plane), a velocity equal to the speed of light must be maintained in the direction $\boldsymbol{\beta}$ of travel. This is referred to as the phase velocity v_{pr} along the direction $\boldsymbol{\beta}$ of travel. Since the energy also is being transported with the same speed, the energy velocity v_{er} in the direction $\boldsymbol{\beta}$ of travel is also equal to the speed of light. Thus

$$v_r = v_{pr} = v_{er} = v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{4-22}$$

where

$v_r =$ wave velocity in the direction of wave travel
$v_{pr} =$ phase velocity in the direction of wave travel
$v_{er} =$ energy (group) velocity in the direction of wave travel
$v =$ speed of light

To keep in step with a constant phase plane of the wave of Figure 4-6, a velocity in the z direction equal to

$$v_{pz} = \frac{v}{\cos\theta_i} = \frac{1}{\sqrt{\mu\varepsilon}\cos\theta_i} \geq v \tag{4-23}$$

must be maintained. This is referred to as the phase velocity v_{pz} in the z direction, and it is greater than the speed of light. Since nothing travels with speeds greater than the speed of light, it must be remembered that this is a hypothetical velocity that must be maintained in order to

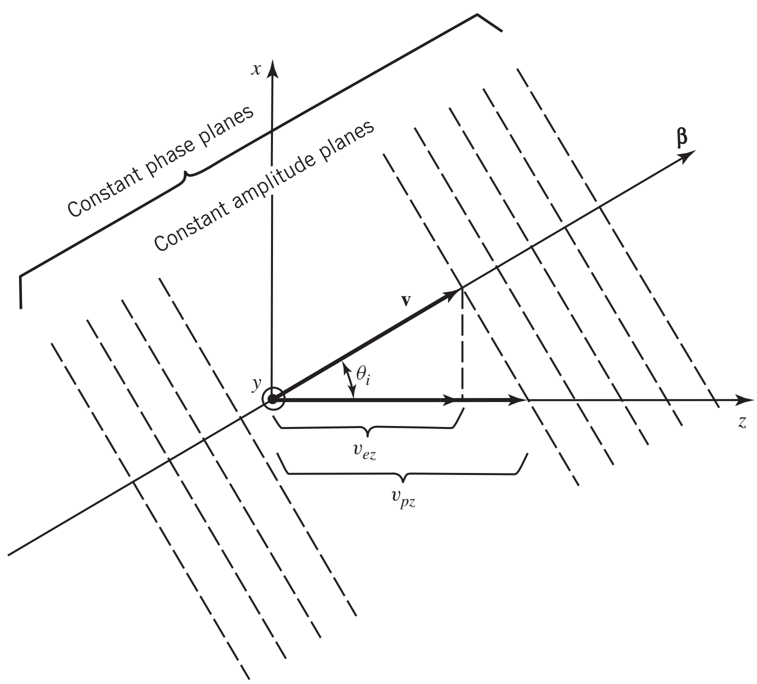


Figure 4-6 Phase and energy (group) velocities of a uniform plane wave.

keep in step with a constant phase plane of the wave that itself travels with the speed of light in the direction $\boldsymbol{\beta}$ of travel. The phase velocities of (4-22) and (4-23) can be obtained, respectively, by using (4-19c) and (4-19d). These are left as end-of-chapter exercises for the reader.

Whereas a velocity greater than the speed of light must be maintained in the z direction to keep in step with a constant phase plane of Figure 4-6, the energy is transported in the z direction with a velocity that is equal to

$$v_{ez} = v\cos\theta_i = \frac{\cos\theta_i}{\sqrt{\mu\varepsilon}} \le v \tag{4-24}$$

This is referred to as the energy (group) velocity v_{ez} in the z direction, and it is equal to or smaller than the speed of light. Graphically this is illustrated in Figure 4-6.

For any wave, the product of the phase and energy velocities in any direction must be equal to the speed of light squared or

$$v_{pr}v_{er} = v_{pz}v_{ez} = v^2 = \frac{1}{\mu\varepsilon} \tag{4-25}$$

This obviously is satisfied by the previously derived results.

The energy velocity of (4-24) can be derived using (4-18a) and (4-18b) along with the definition (4-9). This is left for the reader as an end-of-chapter exercise.

Since the fields of (4-18a) and (4-18b) form a uniform plane wave, the planes over which the amplitude is maintained constant are also constant planes that are perpendicular to the direction $\boldsymbol{\beta}$ of travel. These are illustrated in Figure 4-6 and coincide with the constant phase planes. For other types of waves, the constant phase and amplitude planes do not in general coincide.

D. Power and Energy Densities The average power density associated with the fields of (4-18a) and (4-18b) that travel in the $\boldsymbol{\beta}^+$ direction is given by

$$\begin{aligned}
\left(\mathbf{S}_{\text{av}}^+\right)_r &= \frac{1}{2}\text{Re}\left[\left(\mathbf{E}^+\right)\times\left(\mathbf{H}^+\right)^*\right] \\
&= \frac{1}{2}\text{Re}\left[E_0^+\left(\hat{\mathbf{a}}_x\cos\theta_i - \hat{\mathbf{a}}_z\sin\theta_i\right)e^{-j\beta(x\sin\theta_i + z\cos\theta_i)}\right. \\
&\qquad\qquad \left.\times\hat{\mathbf{a}}_y\frac{E_0^{+*}}{\eta}e^{+j\beta(x\sin\theta_i+z\cos\theta_i)}\right] \\
\left(\mathbf{S}_{\text{av}}^+\right)_r &= \left(\hat{\mathbf{a}}_x\sin\theta_i + \hat{\mathbf{a}}_z\cos\theta_i\right)\frac{|E_0^+|^2}{2\eta} = \hat{\mathbf{a}}_r\frac{|E_0^+|^2}{2\eta} = \hat{\mathbf{a}}_x\left(S_{\text{av}}^+\right)_x + \hat{\mathbf{a}}_z\left(S_{\text{av}}^+\right)_z
\end{aligned} \tag{4-26}$$

where

$$\left(S_{\text{av}}^+\right)_x = \sin\theta_i\frac{|E_0^+|^2}{2\eta} = \sin\theta_i\left(S_{\text{av}}^+\right)_r \tag{4-26a}$$

$$\left(S_{\text{av}}^+\right)_z = \cos\theta_i\frac{|E_0^+|^2}{2\eta} = \cos\theta_i\left(S_{\text{av}}^+\right)_r \tag{4-26b}$$

$(S_{\text{av}}^+)_r$ represents the average power density along the principal $\boldsymbol{\beta}^+$ direction of travel and $(S_{\text{av}}^+)_x$ and $(S_{\text{av}}^+)_z$ represent the directional power densities of the wave in the $+x$ and $+z$ directions, respectively. Similar expressions can be derived for the wave that travels along the $\boldsymbol{\beta}^-$ direction.

Example 4-3

For the TM^y fields of Example 4-2, derive expressions for the average power density along the principal $\boldsymbol{\beta}^+$ direction of travel and for the directional power densities along the $+x$ and $+z$ directions.

Solution: Using the electric and magnetic fields of the solution of Example 4-2 and following the procedure used to derive (4-26) through (4-26b), it can be shown that

$$\left(\mathbf{S}_{\text{av}}^+\right)_r = \frac{1}{2}\text{Re}\left[\left(\mathbf{E}^+\right) \times \left(\mathbf{H}^+\right)^*\right]$$
$$= \frac{1}{2}\text{Re}\left[\hat{\mathbf{a}}_y E_0^+ e^{-j\beta(x\cos\theta_i + y\sin\theta_i)} \times \frac{\left(E_0^+\right)^*}{\eta}\left(-\hat{\mathbf{a}}_x \cos\theta_i + \hat{\mathbf{a}}_z \sin\theta_i\right) e^{+j\beta(x\cos\theta_i + y\sin\theta_i)}\right]$$

$$\left(\mathbf{S}_{\text{av}}^+\right)_r = \left(\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i\right)\frac{|E_0^+|^2}{2\eta} = \hat{\mathbf{a}}_r \frac{|E_0^+|^2}{2\eta}$$
$$= \hat{\mathbf{a}}_x \left(S_{\text{av}}^+\right)_x + \hat{\mathbf{a}}_z \left(S_{\text{av}}^+\right)_z$$

where

$$\left(S_{\text{av}}^+\right)_x = \sin\theta_i \frac{|E_0^+|^2}{2\eta} = \sin\theta_i \left(S_{\text{av}}^+\right)_r$$
$$\left(S_{\text{av}}^+\right)_z = \cos\theta_i \frac{|E_0^+|^2}{2\eta} = \cos\theta_i \left(S_{\text{av}}^+\right)_r$$

(S_{av}^+), $(S_{\text{av}}^+)_x$, and $(S_{\text{av}}^+)_z$ of this TM^y wave are identical to the corresponding ones of the TE^y wave, given by (4-26) through (4-26b).

4.3 TRANSVERSE ELECTROMAGNETIC MODES IN LOSSY MEDIA

In addition to the accumulation of phase, electromagnetic waves that travel in lossy media undergo attenuation. To account for the attenuation, an attenuation constant is introduced as discussed in Chapter 3, Section 3.4.1B. In this section we want to discuss the solution for the electric and magnetic fields of uniform plane waves as they travel in lossy media [5].

4.3.1 Uniform Plane Waves in an Unbounded Lossy Medium — Principal Axis

As for the electromagnetic wave of Section 4.2.1, let us assume that a uniform plane wave is traveling in a lossy medium. Using the coordinate system of Figure 4-1, the electric field is assumed to have an x component and the wave is traveling in the $\pm z$ direction. Since the electric field must satisfy the wave equation for lossy media, its expression takes, according to (3-42a), the form

$$\mathbf{E}(z) = \hat{\mathbf{a}}_x E_x(z) = \hat{\mathbf{a}}_x\left(E_0^+ e^{-\gamma z} + E_0^- e^{+\gamma z}\right) = \hat{\mathbf{a}}_x\left(E_0^+ e^{-\alpha z} e^{-j\beta z} + E_0^- e^{+\alpha z} e^{+j\beta z}\right) \tag{4-27}$$

where $\gamma_x = \gamma_y = 0$ and $\gamma_z = \gamma$. The first term represents the positive traveling wave and the second term represents the negative traveling wave. In (4-27) γ is the propagation constant whose real α and imaginary β parts are defined, respectively, as the attenuation and phase constants. According to (3-37e) and (3-46), γ takes the form

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu\,(\sigma + j\omega\varepsilon)} = \sqrt{-\omega^2\mu\varepsilon + j\omega\mu\sigma} \tag{4-28}$$

Squaring (4-28) and equating real and imaginary from both sides reduces it to

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon \tag{4-28a}$$

$$2\alpha\beta = \omega\mu\sigma \tag{4-28b}$$

Solving (4-28a) and (4-28b) simultaneously, we can write α and β as

$$\boxed{\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]\right\}^{1/2} \quad \text{Np/m}} \tag{4-28c}$$

$$\boxed{\beta = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]\right\}^{1/2} \quad \text{rad/m}} \tag{4-28d}$$

In the literature, the phase constant β is also represented by k.

The attenuation constant α is often expressed in decibels per meter (dB/m). The conversion between Nepers per meter and decibels per meter is obtained by examining the real exponential in (4-27) that represents the attenuation factor of the wave in a lossy medium. Since that factor represents the relative attenuation of the electric or magnetic field, its conversion to decibels (dB) is obtained by

$$\begin{aligned} \text{dB} &= 20\log_{10}\left(e^{-\alpha z}\right) = 20\,(-\alpha z)\log_{10}(e) \\ &= 20\,(-\alpha z)\,(0.434) = -8.68\,(\alpha z) \end{aligned} \tag{4-28e}$$

or

$$|\alpha\ (\text{Np/m})\,| = \frac{1}{8.68}|\alpha\ (\text{dB/m})\,| \tag{4-28f}$$

The magnetic field associated with the electric field of (4-27) can be obtained using Maxwell's equation 4-3 or 4-3a, that is,

$$\mathbf{H} = -\frac{1}{j\omega\mu}\nabla \times \mathbf{E} = -\hat{\mathbf{a}}_y \frac{1}{j\omega\mu}\frac{\partial E_x}{\partial z} \tag{4-29}$$

Using (4-27) reduces (4-29) to

$$\begin{aligned} \mathbf{H} &= +\hat{\mathbf{a}}_y \frac{\gamma}{j\omega\mu}\left(E_0^+ e^{-\gamma z} - E_0^- e^{+\gamma z}\right) \\ &= \hat{\mathbf{a}}_y \frac{\sqrt{j\omega\mu\,(\sigma + j\omega\varepsilon)}}{j\omega\mu}\left(E_0^+ e^{-\gamma z} - E_0^- e^{+\gamma z}\right) \\ &= \hat{\mathbf{a}}_y \sqrt{\frac{\sigma + j\omega\varepsilon}{j\omega\mu}}\left(E_0^+ e^{-\gamma z} - E_0^- e^{+\gamma z}\right) \end{aligned}$$

$$\mathbf{H} = \hat{\mathbf{a}}_y \frac{1}{Z_w}\left(E_0^+ e^{-\gamma z} - E_0^- e^{+yz}\right) \tag{4-29a}$$

In (4-29a), Z_w is the wave impedance of the wave, and it takes the form

$$Z_w = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \eta_c \tag{4-30}$$

which is also equal to the intrinsic impedance η_c of the lossy medium. The equality between the wave and intrinsic impedances for TEM waves in lossy media is identical to that for lossless media of Section 4.2.1B.

The average power density associated with the positive traveling fields of (4-27) and (4-29a) can be written as

$$\mathbf{S}^+ = \frac{1}{2}\text{Re}\left(\mathbf{E}^+ \times \mathbf{H}^{+*}\right) = \frac{1}{2}\text{Re}\left(\hat{\mathbf{a}}_x E_0^+ e^{-\alpha z} e^{-j\beta z} \times \hat{\mathbf{a}}_y \frac{E_0^{+*}}{\eta_c^*} e^{-\alpha z} e^{+j\beta z}\right)$$

$$\mathbf{S}^+ = \hat{\mathbf{a}}_z \frac{|E_0^+|^2}{2} e^{-2\alpha z} \text{Re}\left[\frac{1}{\eta_c^*}\right] \tag{4-31}$$

Individually each term of (4-27) or (4-29a) represents a traveling wave in its respective direction. The magnitude of each term in (4-27) takes the form

$$|E_x^+(z)| = |E_0^+| e^{-\alpha z} \tag{4-32a}$$

$$|E_x^-(z)| = |E_0^-| e^{+\alpha z} \tag{4-32b}$$

which, when plotted for $-\lambda \leq z \leq +\lambda$ and $|\Gamma| = 0.2$ through 1 (in increments of 0.2), take the form shown in Figure 4-7*a*.

Collectively, both terms in each of the fields in (4-27) or (4-29a) represent a standing wave. Using the procedure outlined in Section 4.2.1D, (4-27) can also be written as

$$E_x(z) = \sqrt{\left(E_0^+\right)^2 e^{-2\alpha z} + \left(E_0^-\right)^2 e^{+2\alpha z} + 2E_0^+ E_0^- \cos(2\beta z)} \times \exp\left\{-j\tan^{-1}\left[\frac{E_0^+ e^{-\alpha z} - E_0^- e^{+\alpha z}}{E_0^+ e^{-\alpha z} + E_0^- e^{+\alpha z}} \tan(\beta z)\right]\right\} \tag{4-33}$$

The standing wave pattern is given by the amplitude term of

$$|E_x(z)| = \sqrt{\left(E_0^+\right)^2 e^{-2\alpha z} + \left(E_0^-\right)^2 e^{+2\alpha z} + 2E_0^+ E_0^- \cos(2\beta z)} \tag{4-33a}$$

which for $|\Gamma| = E_0^-/E_0^+ = 0.2$ through 1, in increments of 0.2, is shown plotted in Figure 4-7*b* in the range $-\lambda \leq z \leq \lambda$ when $f = 100$ MHz, $\varepsilon_r = 2.56$, $\mu_r = 1$, and $\sigma = 0.03$ S/m.

The distance the wave must travel in a lossy medium to reduce its value to $e^{-1} = 0.368 = 36.8\%$ *is defined as the skin depth* δ. For each of the terms of (4-27) or (4-29a), this distance is

$$\boxed{\delta = \text{skin depth} = \frac{1}{\alpha} = \frac{1}{\omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1 + (\sigma/\omega\varepsilon)^2} - 1\right]\right\}^{1/2}} \text{ m}} \tag{4-34}$$

In summary, the attenuation constant α, phase constant β, wave Z_w and intrinsic η_c impedances, wavelength λ, velocity v, and skin depth δ for a uniform plane wave traveling in a lossy medium are listed in the second column of Table 4-1. The same expressions are valid for plane and TEM waves. Simpler expressions for each can be derived depending upon the value of the $(\sigma/\omega\varepsilon)^2$ ratio. Media whose $(\sigma/\omega\varepsilon)^2$ is much less than unity $[(\sigma/\omega\varepsilon)^2 \ll 1]$ are referred to as *good dielectrics* and those whose $(\sigma/\omega\varepsilon)^2$ is much greater than unity $[(\sigma/\omega\varepsilon)^2 \gg 1]$ are referred to as *good conductors* [6]; each will now be discussed.

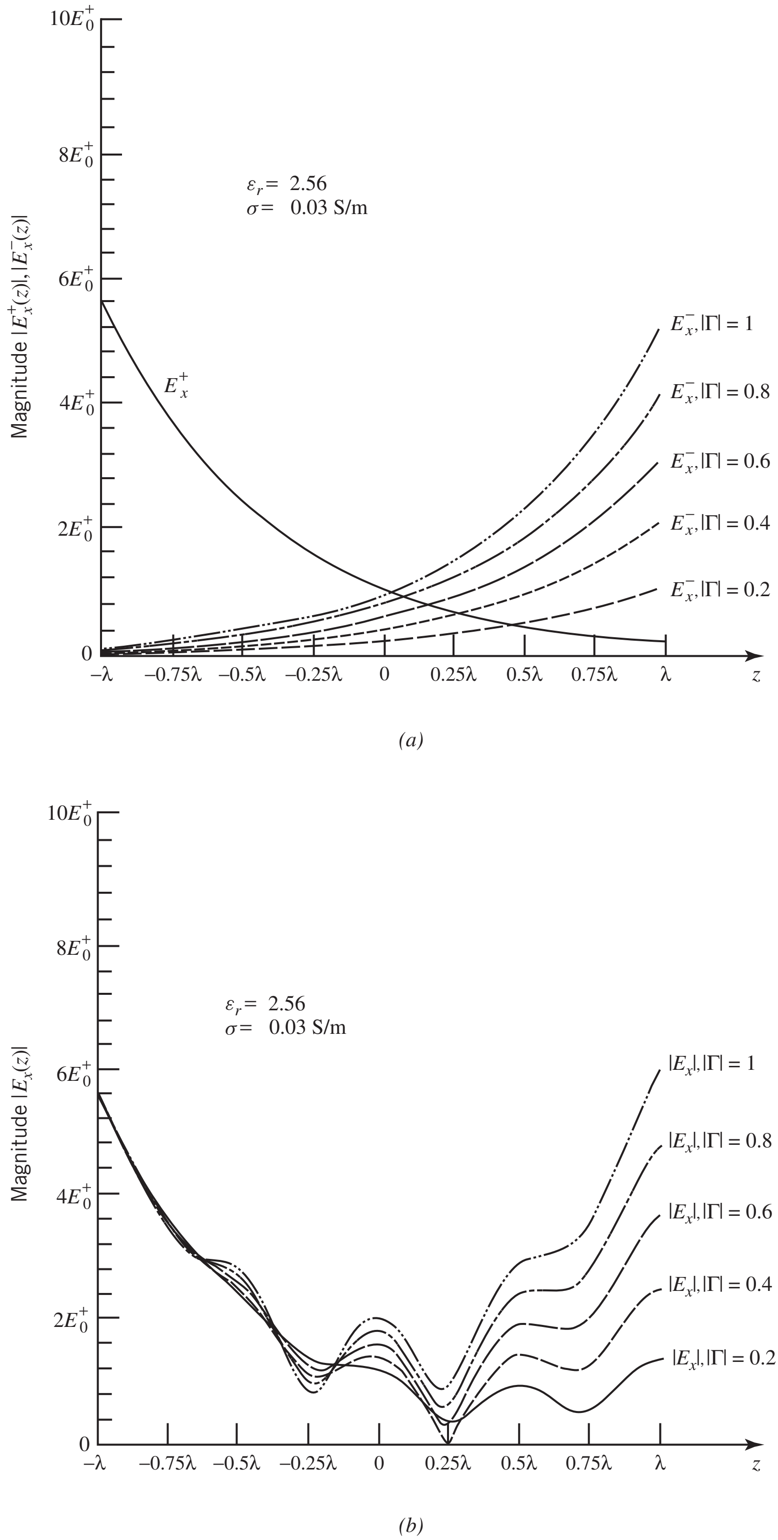


Figure 4-7 Wave patterns of uniform plane waves in a lossy medium. (*a*) Traveling. (*b*) Standing.

TABLE 4-1 Propagation constant, wave impedance, wavelength, velocity, and skin depth of TEM wave in lossy media

	Exact	Good dielectric $\left(\frac{\sigma}{\omega\varepsilon}\right)^2 \ll 1$	Good conductor $\left(\frac{\sigma}{\omega\varepsilon}\right)^2 \gg 1$
Attenuation constant α	$= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]\right\}^{1/2}$	$\simeq \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\simeq \sqrt{\frac{\omega\mu\sigma}{2}}$
Phase constant β	$= \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]\right\}^{1/2}$	$\simeq \omega\sqrt{\mu\varepsilon}$	$\simeq \sqrt{\frac{\omega\mu\sigma}{2}}$
Wave Z_w intrinsic η_c impedances $Z_w = \eta_c$	$= \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$	$\simeq \sqrt{\frac{\mu}{\varepsilon}}$	$\simeq \sqrt{\frac{\omega\mu}{2\sigma}}(1+j)$
Wavelength λ	$= \frac{2\pi}{\beta}$	$\simeq \frac{2\pi}{\omega\sqrt{\mu\varepsilon}}$	$\simeq 2\pi\sqrt{\frac{2}{\omega\mu\sigma}}$
Velocity v	$= \frac{\omega}{\beta}$	$\simeq \frac{1}{\sqrt{\mu\varepsilon}}$	$\simeq \sqrt{\frac{2\omega}{\mu\sigma}}$
Skin depth δ	$= \frac{1}{\alpha}$	$\simeq \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$	$\simeq \sqrt{\frac{2}{\omega\mu\sigma}}$

***A. Good Dielectrics* $[(\sigma/\omega\varepsilon)^2 \ll 1]$** For source-free lossy media, Maxwell's equation in differential form as derived from Ampere's law takes the form, by referring to Table 1-4, of

$$\nabla \times \mathbf{H} = \mathbf{J}_c + \mathbf{J}_d = \sigma\mathbf{E} + j\omega\varepsilon\mathbf{E} = (\sigma + j\omega\varepsilon)\,\mathbf{E} \tag{4-35}$$

where $\mathbf{J}_c$ and $\mathbf{J}_d$ represent, respectively, the conduction and displacement current densities. When $\sigma/\omega\varepsilon \ll 1$, the displacement current density is much greater than the conduction current density; when $\sigma/\omega\varepsilon \gg 1$ the conduction current density is much greater than the displacement current density. For each of these two cases, the exact forms of the field parameters of Table 4-1 can be approximated by simpler forms. This will be demonstrated next.

For a good dielectric [when $(\sigma/\omega\varepsilon)^2 \ll 1$], the exact expression for the attenuation constant of (4-28c) can be written using the binomial expansion and it takes the form

$$\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]\right\}^{1/2}$$

$$\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\left(1+\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2-\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^4\cdots\right)-1\right]\right\}^{1/2} \tag{4-36}$$

Retaining only the first two terms of the infinite series, (4-36) can be approximated by

$$\alpha \simeq \omega\sqrt{\mu\varepsilon}\left[\frac{1}{4}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/2} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \tag{4-36a}$$

In a similar manner it can be shown that by following the same procedure but only retaining the first term of the infinite series, the exact expression for β of (4-28d) can be approximated by

$$\beta \simeq \omega\sqrt{\mu\varepsilon} \tag{4-37}$$

For good dielectrics, the wave and intrinsic impedances of (4-30) can be approximated by

$$Z_w = \eta_c = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu/j\omega\varepsilon}{\sigma/j\omega\varepsilon + 1}} \simeq \sqrt{\frac{\mu}{\varepsilon}} \tag{4-38}$$

while the skin depth can be represented by

$$\delta = \frac{1}{\alpha} \simeq \frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}} \tag{4-39}$$

These and other approximate forms for the parameters of good dielectrics are summarized on the third column of Table 4-1.

***B. Good Conductors* [$(\sigma/\omega\varepsilon)^2 \gg 1$]** For good conductors, the exact expression for the attenuation constant of (4-28c) can be written using the binomial expansion and takes the form

$$\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{\left(\frac{\sigma}{\omega\varepsilon}\right)^2 + 1} - 1\right]\right\}^{1/2} = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\frac{\sigma}{\omega\varepsilon}\left(1 + \frac{1}{(\sigma/\omega\varepsilon)^2}\right)^{1/2} - 1\right]\right\}^{1/2}$$

$$\alpha = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\frac{\sigma}{\omega\varepsilon} + \frac{1}{2}\frac{1}{\sigma/\omega\varepsilon} - \frac{1}{8}\frac{1}{(\sigma/\omega\varepsilon)^3} + \cdots - 1\right]\right\}^{1/2} \tag{4-40}$$

Retaining only the first term of the infinite series expansion, (4-40) can be approximated by

$$\alpha \simeq \omega\sqrt{\mu\varepsilon}\left(\frac{1}{2}\frac{\sigma}{\omega\varepsilon}\right)^{1/2} = \sqrt{\frac{\omega\mu\sigma}{2}} \tag{4-40a}$$

Following a similar procedure, the phase constant of (4-28d) can be approximated by

$$\beta \simeq \sqrt{\frac{\omega\mu\sigma}{2}} \tag{4-41}$$

which is identical to the approximate expression for the attenuation constant of (4-40a).

For good conductors, the wave and intrinsic impedances of (4-30) can be approximated by

$$Z_w = \eta_c = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j\omega\mu/\omega\varepsilon}{\sigma/\omega\varepsilon + j}} \simeq \sqrt{j\frac{\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{2\sigma}}(1 + j) \tag{4-42}$$

whose real and imaginary parts are identical. For the same conditions, the skin depth can be approximated by

$$\delta = \frac{1}{\alpha} \simeq \sqrt{\frac{2}{\omega\mu\sigma}} \tag{4-43}$$

This is the most widely recognized form for the skin depth.

4.3.2 Uniform Plane Waves in an Unbounded Lossy Medium — Oblique Angle

For lossy media the difference between principal axes propagation and propagation at oblique angles is that the propagation constant γ_r along the direction $\boldsymbol{\beta}$ of propagation must be decomposed into its directional components along the principal axes of the coordinate system. In addition, since the propagation constant γ has real (α) and imaginary (β) parts, constant amplitude and constant phase planes are associated with the wave. As discussed in Section 4.2.2C and illustrated in Figure 4-6, the constant phase planes for a uniform plane wave are planes that are parallel to each other, perpendicular to the direction of propagation, and coincide with the constant amplitude planes. The constant amplitude planes are planes over which the amplitude remains constant. For a uniform plane wave traveling in a lossy medium, the constant amplitude planes are also parallel to each other, are perpendicular to the direction of travel, and coincide with the constant phase planes. This is illustrated in Figure 4-6 for a uniform plane wave traveling at an oblique angle in a lossless medium.

Let us assume that a uniform plane wave that is also TE^y is traveling in a lossy medium at an angle θ_i, as shown in Figure 4-4a. Following a procedure similar to the lossless case and referring to (4-17a) and (4-17b), the propagation constant of (4-28) can now be written for the positive and negative traveling waves as

$$\boldsymbol{\gamma}^+ = \gamma\,(\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i) = (\alpha + j\beta)\,(\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i) \tag{4-44a}$$

$$\boldsymbol{\gamma}^- = -\gamma\,(\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i) = -\,(\alpha + j\beta)\,(\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i) \tag{4-44b}$$

where the real (α) and imaginary (β) parts of γ are given by (4-28c) and (4-28d), respectively. Using (4-44a) and (4-44b), the electric and magnetic fields can be written, by referring to (4-17) through (4-18c), as

$$\begin{aligned}
\mathbf{E} &= E_0^+\,(\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\,e^{-\boldsymbol{\gamma}^+\cdot\mathbf{r}} + E_0^-\,(\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\,e^{-\boldsymbol{\gamma}^-\cdot\mathbf{r}} \\
\mathbf{E} &= E_0^+\,(\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\,e^{-(\alpha+j\beta)(x\sin\theta_i + z\cos\theta_i)} \\
&\quad + E_0^-\,(\hat{\mathbf{a}}_x \cos\theta_i - \hat{\mathbf{a}}_z \sin\theta_i)\,e^{+(\alpha+j\beta)(x\sin\theta_i + z\cos\theta_i)}
\end{aligned} \tag{4-45a}$$

$$\mathbf{H} = \hat{\mathbf{a}}_y \left[\frac{E_0^+}{\eta_c} e^{-(\alpha+j\beta)(x\sin\theta_i + z\cos\theta_i)} - \frac{E_0^-}{\eta_c} e^{+(\alpha+j\beta)(x\sin\theta_i + z\cos\theta_i)}\right] \tag{4-45b}$$

Because the wave is a uniform plane wave in the $\boldsymbol{\beta}$ direction of propagation, the wave impedance Z_{wr} in the direction of propagation is equal to the intrinsic impedance η_c of the lossy medium given by (4-30) or

$$Z_{wr} = \eta_c = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \tag{4-46}$$

However, the directional impedances in the x and z directions are given, by referring to (4-20a) and (4-20b), by

$$Z_x^+ = -\frac{E_z^+}{H_y^+} = \eta_c \sin\theta_i = Z_x^- = \frac{E_z^-}{H_y^-} \tag{4-47a}$$

$$Z_z^+ = \frac{E_x^+}{H_y^+} = \eta_c \cos\theta_i = Z_z^- = -\frac{E_x^-}{H_y^-} \tag{4-47b}$$

According to (4-22) through (4-24) the phase and energy velocities in the principal $\boldsymbol{\beta}$ direction of travel and in the z direction are given, respectively, by

$$v_r = v_{pr} = v_{er} = v = \frac{\omega}{\beta} \tag{4-48a}$$

$$v_{pz} = \frac{v}{\cos\theta_i} = \frac{\omega}{\beta\cos\theta_i} \geq v = \frac{\omega}{\beta} \tag{4-48b}$$

$$v_{ez} = v\cos\theta_i = \frac{\omega}{\beta}\cos\theta_i \leq v = \frac{\omega}{\beta} \tag{4-48c}$$

where β for a lossy medium is given by (4-28d) or

$$\beta = \omega\sqrt{\mu\varepsilon}\left\{\frac{1}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]\right\}^{1/2} \tag{4-48d}$$

As for the lossless medium, the product of the phase and energy velocities is equal to the square of the velocity of light v in the lossy medium, or

$$v_{pr}v_{er} = v_{pz}v_{ez} = v^2 \tag{4-48e}$$

Using the procedure followed to derive (4-26) through (4-26b) and (4-31), the average power density along the principal direction $\boldsymbol{\beta}$ of travel and the directional power densities along the x and z directions can be written for the fields of (4-45a) and (4-45b) as

$$\begin{aligned}\left(\mathbf{S}_{\text{av}}^{+}\right)_r &= (\hat{\mathbf{a}}_x \sin\theta_i + \hat{\mathbf{a}}_z \cos\theta_i)\frac{\left|E_0^+\right|^2}{2}e^{-2\alpha(x\sin\theta_i + z\cos\theta_i)}\text{Re}\left[\frac{1}{\eta_c^*}\right] \\ &= \hat{\mathbf{a}}_r\frac{\left|E_0^+\right|^2}{2}e^{-2\alpha r}\text{Re}\left[\frac{1}{\eta_c^*}\right]\end{aligned} \tag{4-49a}$$

$$\begin{aligned}\left(S_{\text{av}}^{+}\right)_x &= \sin\theta_i\frac{\left|E_0^+\right|^2}{2}e^{-2\alpha(x\sin\theta_i + z\cos\theta_i)}\text{Re}\left[\frac{1}{\eta_c^*}\right] \\ &= \sin\theta_i\frac{\left|E_0^+\right|^2}{2}e^{-2\alpha r}\text{Re}\left[\frac{1}{\eta_c^*}\right]\end{aligned} \tag{4-49b}$$

$$\begin{aligned}\left(S_{\text{av}}^{+}\right)_z &= \cos\theta_i\frac{\left|E_0^+\right|^2}{2}e^{-2\alpha(x\sin\theta_i + z\cos\theta_i)}\text{Re}\left[\frac{1}{\eta_c^*}\right] \\ &= \cos\theta_i\frac{\left|E_0^+\right|^2}{2}e^{-2\alpha r}\text{Re}\left[\frac{1}{\eta_c^*}\right]\end{aligned} \tag{4-49c}$$

Example 4-4

For a TM^y wave traveling in a lossy medium at an oblique angle θ_i, derive expressions for the fields, wave impedances, phase and energy velocities, and average power densities.

Solution: The solution to this problem can be accomplished by following the procedure used to derive the expressions of the fields and other wave characteristics of a TE^y wave traveling at an oblique angle in a lossy medium, as outlined in this section, and referring to the solution of Examples 4-2 and 4-3. Doing this we can write the fields of a TM^y traveling in a lossy medium at an oblique angle θ_i, the coordinate system of which is illustrated in Figure 4-4*b*, as

$$\mathbf{E} = \mathbf{E}^+ + \mathbf{E}^- = \hat{\mathbf{a}}_y\left[E_0^+e^{-(\alpha+j\beta)(x\sin\theta_i + z\cos\theta_i)} + E_0^-e^{+(\alpha+j\beta)(x\sin\theta_i + z\cos\theta_i)}\right]$$

$$\begin{aligned}\mathbf{H} = \mathbf{H}^+ + \mathbf{H}^- = {}& \frac{E_0^+}{\eta_c}(-\hat{\mathbf{a}}_x\cos\theta_i + \hat{\mathbf{a}}_z\sin\theta_i)\,e^{-(\alpha+j\beta)(x\sin\theta_i + z\cos\theta_i)} \\ & + \frac{E_0^-}{\eta_c}(\hat{\mathbf{a}}_x\cos\theta_i - \hat{\mathbf{a}}_z\sin\theta_i)\,e^{+(\alpha+j\beta)(x\sin\theta_i + z\cos\theta_i)}\end{aligned}$$

In addition, the wave impedances are given, by referring to (4-21a) and (4-21b), by

$$Z_x^+ = \frac{E_y^+}{H_z^+} = \frac{\eta_c}{\sin\theta_i} = Z_x^- = -\frac{E_y^-}{H_z^-}$$

$$Z_z^+ = -\frac{E_y^+}{H_x^+} = \frac{\eta_c}{\cos\theta_i} = Z_z^- = \frac{E_y^-}{H_x^-}$$

The phase and energy velocities, and their relationships, are the same as those for the TE^y wave, as given by (4-48a) through (4-48e). Similarly, the average power densities are those given by (4-49a) through (4-49c).

4.4 POLARIZATION

According to the *IEEE Standard Definitions for Antennas* [7, 8], the *polarization of a radiated wave* is defined as "that property of a radiated electromagnetic wave describing the time-varying direction and relative magnitude of the electric field vector; specifically, the figure traced as a function of time by the extremity of the vector at a fixed location in space, and the sense in which it is traced, as observed along the direction of propagation." In other words, polarization is the curve traced out, at a given observation point as a function of time, by the end point of the arrow representing the instantaneous electric field. The field must be observed along the direction of propagation. A typical trace as a function of time is shown in Figure 4-8 [8].

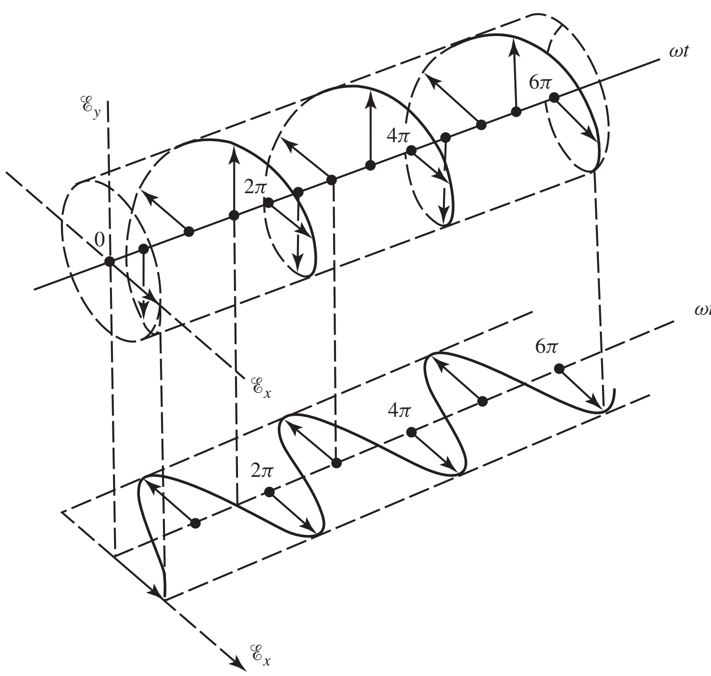


Figure 4-8 Rotation of a plane electromagnetic wave at $z = 0$ as a function of time. (Source: C. A. Balanis, *Antenna Theory: Analysis and Design*. 3rd Edition. Copyright © 2005, John Wiley & Sons, Inc. Reprinted by permission of John Wiley & Sons, Inc.).

Polarization may be classified into three categories: *linear, circular, and elliptical* [8]. If the vector that describes the electric field at a point in space as a function of time is always directed along a line, which is normal to the direction of propagation, the field is said to be *linearly* polarized. In general, however, the figure that the electric field traces is an ellipse, and the field is said to be *elliptically* polarized. Linear and circular polarizations are special cases of elliptical, and they can be obtained when the ellipse becomes a straight line or a circle, respectively. The figure of the electric field is traced in a *clockwise* (CW) or *counterclockwise* (CCW) sense. Clockwise rotation of the electric field vector is also designated as *right-hand polarization* and counterclockwise as *left-hand polarization*. In Figure 4-9 we show the figure traced by the extremity of the time-varying field vector for linear, circular, and elliptical polarizations.

The mathematical details for defining linear, circular, and elliptical polarizations follow.

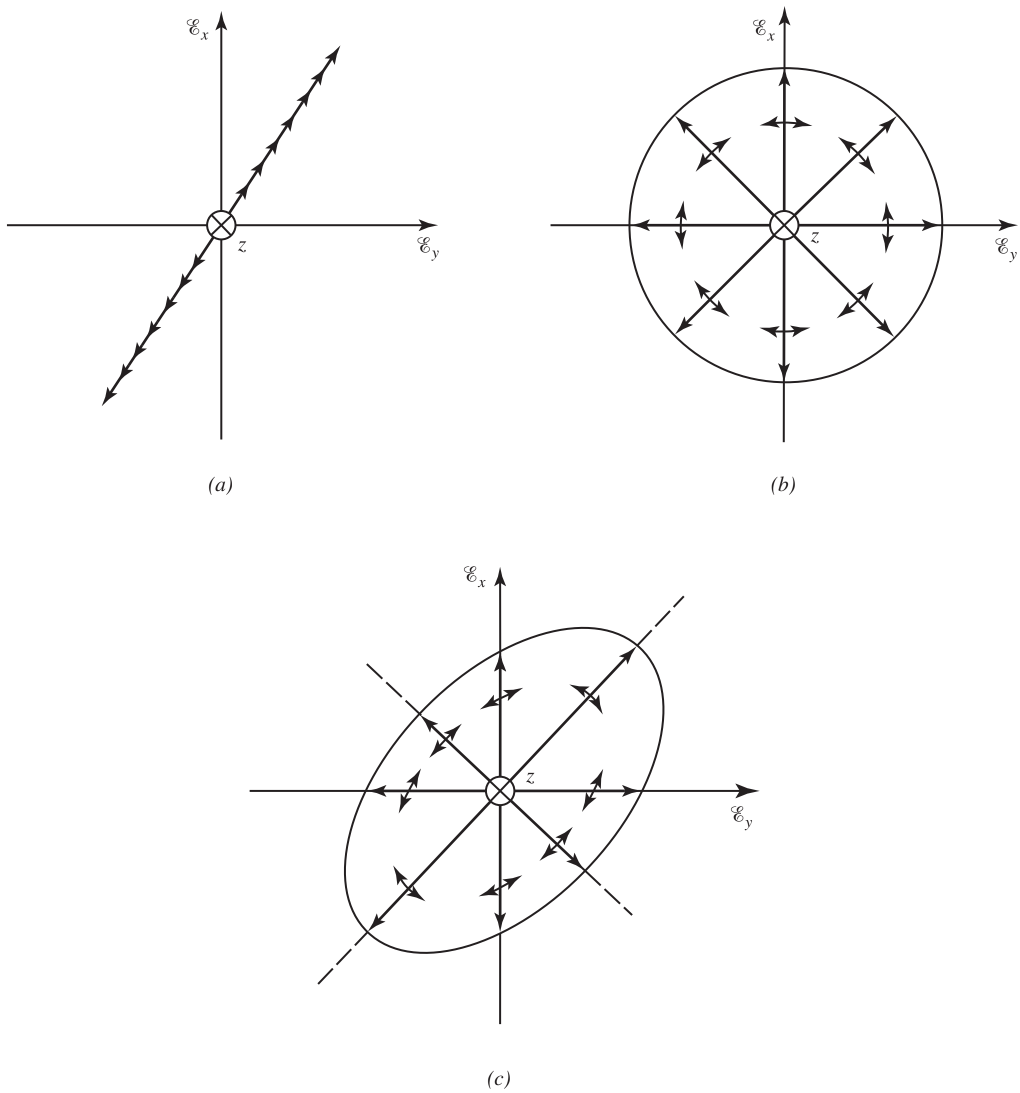


Figure 4-9 Polarization figure traces of an electric field extremity *as a function of time for a fixed position*. (*a*) Linear. (*b*) Circular. (*c*) Elliptical.

4.4.1 Linear Polarization

Let us consider a harmonic plane wave, with x and y electric field components, traveling in the positive z direction (into the page), as shown in Figure 4-10 [8]. The instantaneous electric and magnetic fields are given by

$$\begin{aligned}\boldsymbol{\mathscr{E}} &= \hat{\mathbf{a}}_x \mathscr{E}_x + \hat{\mathbf{a}}_y \mathscr{E}_y = \operatorname{Re}\left[\hat{\mathbf{a}}_x E_x^+ e^{j(\omega t-\beta z)} + \hat{\mathbf{a}}_y E_y^+ e^{j(\omega t-\beta z)}\right]\\ &= \hat{\mathbf{a}}_x E_{x_0}^+ \cos(\omega t - \beta z + \phi_x) + \hat{\mathbf{a}}_y E_{y_0}^+ \cos\left(\omega t - \beta z + \phi_y\right)\end{aligned} \tag{4-50a}$$

$$\begin{aligned}\boldsymbol{\mathscr{H}} &= \hat{\mathbf{a}}_y \mathscr{H}_y + \hat{\mathbf{a}}_x \mathscr{H}_x = \operatorname{Re}\left[\hat{\mathbf{a}}_y \frac{E_x^+}{\eta} e^{j(\omega t-\beta z)} - \hat{\mathbf{a}}_x \frac{E_y^+}{\eta} e^{j(\omega t-\beta z)}\right]\\ &= \hat{\mathbf{a}}_y \frac{E_{x_0}^+}{\eta} \cos(\omega t - \beta z + \phi_x) - \hat{\mathbf{a}}_x \frac{E_{y_0}^+}{\eta} \cos\left(\omega t - \beta z + \phi_y\right)\end{aligned} \tag{4-50b}$$

where E_x^+, E_y^+ are complex and $E_{x_0}^+$, $E_{y_0}^+$ are real.

Let us now examine the variation of the instantaneous electric field vector $\mathscr{E}$ as given by (4-50a) at the $z = 0$ plane. Other planes may be considered, but the $z = 0$ plane is chosen for convenience and simplicity. For the first example, let

$$E_{y_0}^+ = 0 \tag{4-51}$$

in (4-50a). Then

$$\begin{aligned}\mathscr{E}_x &= E_{x_0}^+ \cos(\omega t + \phi_x)\\ \mathscr{E}_y &= 0\end{aligned} \tag{4-51a}$$

The locus of the instantaneous electric field vector is given by

$$\boldsymbol{\mathscr{E}} = \hat{\mathbf{a}}_x E_{x_0}^+ \cos(\omega t + \phi_x) \tag{4-51b}$$

which is a straight line, and it will always be directed along the x axis at all times, as shown in Figure 4-10. The field is said to be *linearly polarized in the x direction*.

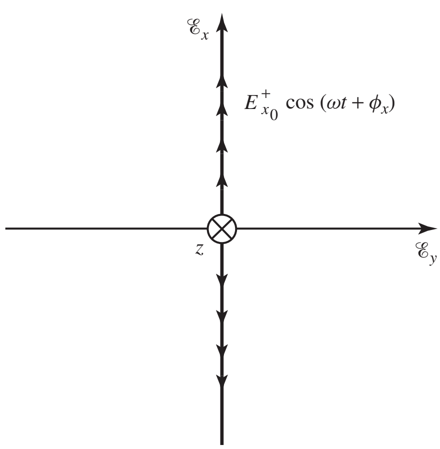


Figure 4-10 Linearly polarized field in the x direction.

Example 4-5

Determine the polarization of the wave given by (4-50a) when $E_{x0}^{+} = 0$.

Solution: Since

$$E_{x0}^{+} = 0$$

then

$$\mathscr{E}_x = 0$$
$$\mathscr{E}_y = E_{y0}^{+} \cos\left(\omega t + \phi_y\right)$$

The locus of the instantaneous electric field vector is given by

$$\boldsymbol{\mathscr{E}} = \hat{\mathbf{a}}_y E_{y0}^{+} \cos\left(\omega t + \phi_y\right)$$

which again is a straight line but directed along the y axis at all times, as shown in Figure 4-11. The field is said to be *linearly polarized in the y direction*.

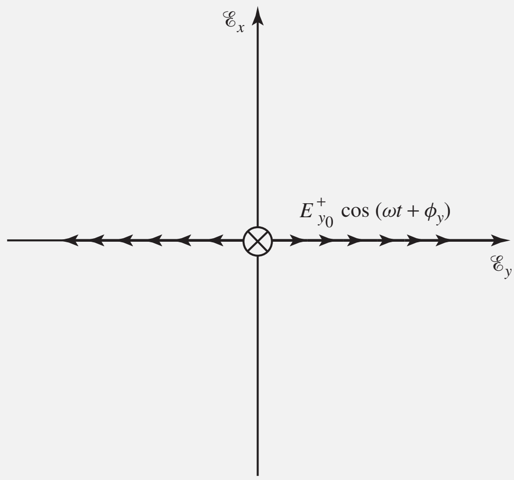


Figure 4-11 Linearly polarized field in the y direction.

Example 4-6

Determine the polarization and direction of polarization of the wave given by (4-50a) when $\phi_x = \phi_y = \phi$.

Solution: Since

$$\phi_x = \phi_y = \phi$$

then

$$\mathscr{E}_x = E_{x0}^{+} \cos(\omega t + \phi)$$
$$\mathscr{E}_y = E_{y0}^{+} \cos(\omega t + \phi)$$

The amplitude of the electric field vector is given by

$$\mathscr{E} = \sqrt{\mathscr{E}_x^2 + \mathscr{E}_y^2} = \sqrt{\left(E_{x0}^+\right)^2 + \left(E_{y0}^+\right)^2} \cos(\omega t + \phi)$$

which is a straight line directed at all times along a line that makes an angle ψ with the x axis as shown in Figure 4-12. The angle ψ is given by

$$\psi = \tan^{-1}\left[\frac{\mathscr{E}_y}{\mathscr{E}_x}\right] = \tan^{-1}\left[\frac{E_{y0}^+}{E_{x0}^+}\right]$$

The field is said to be *linearly polarized in the ψ direction*.

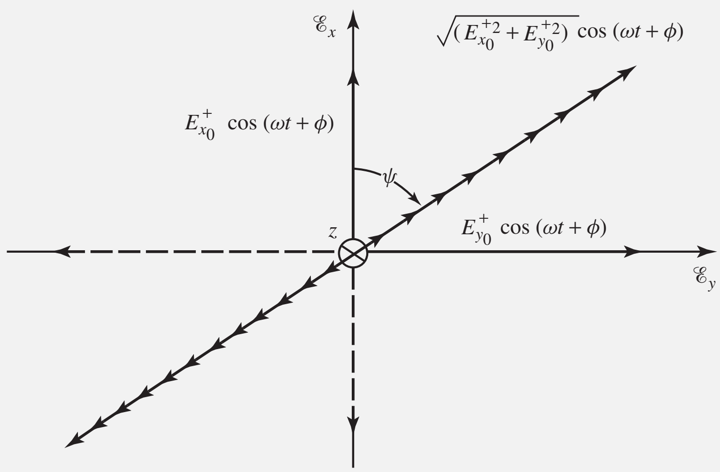


Figure 4-12 Linearly polarized field in the ψ direction.

It is evident from the preceding examples that *a time-harmonic field is linearly polarized at a given point in space if the electric field (or magnetic field) vector at that point is oriented along the same straight line at every instant of time*. This is accomplished if the field vector (electric or magnetic) possesses (a) only one component or (b) two orthogonal linearly polarized components that are in time phase or integer multiples of 180° out of phase.

4.4.2 Circular Polarization

A wave is said to be *circularly polarized if the tip of the electric field vector traces out a circular locus in space*. At various instants of time, the electric field intensity of such a wave always has the same amplitude and the orientation in space of the electric field vector changes continuously with time in such a manner as to describe a circular locus [8, 9].

A. Right-Hand (Clockwise) Circular Polarization A wave has *right-hand circular polarization* if its electric field vector has a clockwise sense of rotation *when it is viewed along the axis of propagation*. In addition, the electric field vector must trace a circular locus if the wave is to have also a circular polarization.

Let us examine the locus of the instantaneous electric field vector ($\boldsymbol{\mathscr{E}}$) at the $z = 0$ plane at all times. For this particular example, let in (4-50a)

$$\begin{aligned} \phi_x &= 0 \\ \phi_y &= -\pi/2 \\ E_{x0}^+ &= E_{y0}^+ = E_R \end{aligned} \tag{4-52}$$

Then

$$\begin{aligned} \mathscr{E}_x &= E_R \cos(\omega t) \\ \mathscr{E}_y &= E_R \cos\left(\omega t - \frac{\pi}{2}\right) = E_R \sin(\omega t) \end{aligned} \tag{4-52a}$$

The locus of the amplitude of the electric field vector is given by

$$\mathscr{E} = \sqrt{\mathscr{E}_x^2 + \mathscr{E}_y^2} = \sqrt{E_R^2(\cos^2 \omega t + \sin^2 \omega t)} = E_R \tag{4-52b}$$

and it is directed along a line making an angle ψ with the x axis, which is given by

$$\psi = \tan^{-1}\left[\frac{\mathscr{E}_y}{\mathscr{E}_x}\right] = \tan^{-1}\left[\frac{E_R \sin(\omega t)}{E_R \cos(\omega t)}\right] = \tan^{-1}[\tan(\omega t)] = \omega t \tag{4-52c}$$

If we plot the locus of the electric field vector for various times at the $z = 0$ plane, we see that it forms a circle of radius E_R and it rotates clockwise with an angular frequency ω, as shown in Figure 4-13. Thus the wave is said to have a *right-hand circular polarization*. Remember that the rotation is viewed from the "rear" of the wave in the direction of propagation. In this example, the wave is traveling in the positive z direction (into the page) so that the rotation is examined from an observation point looking into the page and perpendicular to it.

We can write the instantaneous electric field vector as

$$\begin{aligned} \boldsymbol{\mathscr{E}} &= \text{Re}\left[\hat{\mathbf{a}}_x E_R e^{j(\omega t - \beta z)} + \hat{\mathbf{a}}_y E_R e^{j(\omega t - \beta z - \pi/2)}\right] \\ &= E_R \text{Re}\left\{\left[\hat{\mathbf{a}}_x - j\hat{\mathbf{a}}_y\right] e^{j(\omega t - \beta z)}\right\} \end{aligned} \tag{4-52d}$$

We note that there is a 90° phase difference between the two orthogonal components of the electric field vector.

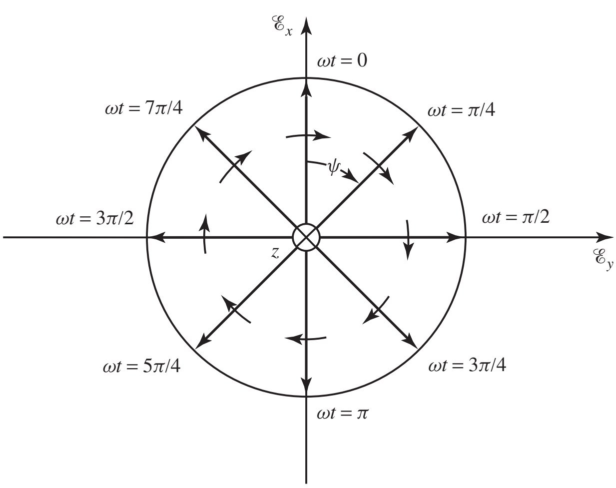


Figure 4-13 Right-hand circularly polarized wave.

Example 4-7

If $\phi_x = +\pi/2$, $\phi_y = 0$, and $E_{x0}^{+} = E_{y0}^{+} = E_{\text{R}}$, determine the polarization and sense of rotation of the wave of (4-50a).

Solution: Since

$$\begin{aligned}\phi_x &= +\frac{\pi}{2}\\ \phi_y &= 0\\ E_{x0}^{+} = E_{y0}^{+} &= E_{\text{R}}\end{aligned}$$

then

$$\begin{aligned}\mathscr{E}_x &= E_{\text{R}}\cos\left(\omega t + \frac{\pi}{2}\right) = -E_{\text{R}}\sin\omega t\\ \mathscr{E}_y &= E_{\text{R}}\cos(\omega t)\end{aligned}$$

and the locus of the amplitude of the electric field vector is given by

$$\mathscr{E} = \sqrt{\mathscr{E}_x^2 + \mathscr{E}_y^2} = \sqrt{E_{\text{R}}^2(\cos^2\omega t + \sin^2\omega t)} = E_{\text{R}}$$

The angle ψ along which the field is directed is given by

$$\psi = \tan^{-1}\left[\frac{\mathscr{E}_y}{\mathscr{E}_x}\right] = \tan^{-1}\left[-\frac{E_{\text{R}}\cos(\omega t)}{E_{\text{R}}\sin(\omega t)}\right] = \tan^{-1}\left[-\cot(\omega t)\right] = \omega t + \frac{\pi}{2}$$

The locus of the field vector is a circle of radius E_{R}, and it rotates clockwise with an angular frequency ω as shown in Figure 4-14; *hence, it is a right-hand circular polarization*.

The expression for the instantaneous electric field vector is

$$\begin{aligned}\boldsymbol{\mathscr{E}} &= \text{Re}\left[\hat{\mathbf{a}}_x E_{\text{R}} e^{j(\omega t - \beta z + \pi/2)} + \hat{\mathbf{a}}_y E_{\text{R}} e^{j(\omega t - \beta z)}\right]\\ &= E_{\text{R}}\text{Re}\left\{\left[j\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y\right] e^{j(\omega t - \beta z)}\right\}\end{aligned}$$

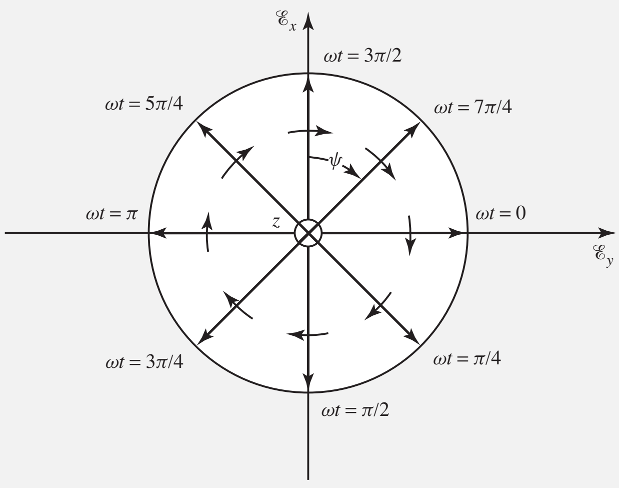


Figure 4-14 Right-hand circularly polarized wave.

Again we note a 90° phase difference between the orthogonal components.

From the previous discussion we see that a right-hand *circular polarization* can be achieved *if and only if its two orthogonal linearly polarized components have equal amplitudes and a* 90° *phase difference of one relative to the other. The sense of rotation* (*clockwise here*) *is determined by rotating the phase-leading component* (*in this instance* $\mathscr{E}_x$) *toward the phase-lagging component* (*in this instance* $\mathscr{E}_y$). *The field rotation must be viewed as the wave travels away from the observer*.

B. Left-Hand (Counterclockwise) Circular Polarization If the electric field vector has a counterclockwise sense of rotation, the polarization is designated as *left-hand polarization*. To demonstrate this, let in (4-50a)

$$\begin{aligned}\phi_x &= 0\\ \phi_y &= \frac{\pi}{2}\\ E_{x_0}^{+} &= E_{y_0}^{+} = E_{\mathrm{L}}\end{aligned} \tag{4-53}$$

then

$$\begin{aligned}\mathscr{E}_x &= E_{\mathrm{L}}\cos(\omega t)\\ \mathscr{E}_y &= E_{\mathrm{L}}\cos\left(\omega t + \frac{\pi}{2}\right) = -E_{\mathrm{L}}\sin(\omega t)\end{aligned} \tag{4-53a}$$

and the locus of the amplitude is

$$\mathscr{E} = \sqrt{\mathscr{E}_x^2 + \mathscr{E}_y^2} = \sqrt{E_{\mathrm{L}}^2(\cos^2\omega t + \sin^2\omega t)} = E_{\mathrm{L}} \tag{4-53b}$$

The angle ψ is given by

$$\psi = \tan^{-1}\left[\frac{\mathscr{E}_y}{\mathscr{E}_x}\right] = \tan^{-1}\left[\frac{-E_{\mathrm{L}}\sin(\omega t)}{E_{\mathrm{L}}\cos(\omega t)}\right] = -\omega t \tag{4-53c}$$

The locus of the field vector is a circle of radius E_{L}, and it rotates counterclockwise with an angular frequency ω as shown in Figure 4-15; *hence, it is a left-hand circular polarization*.

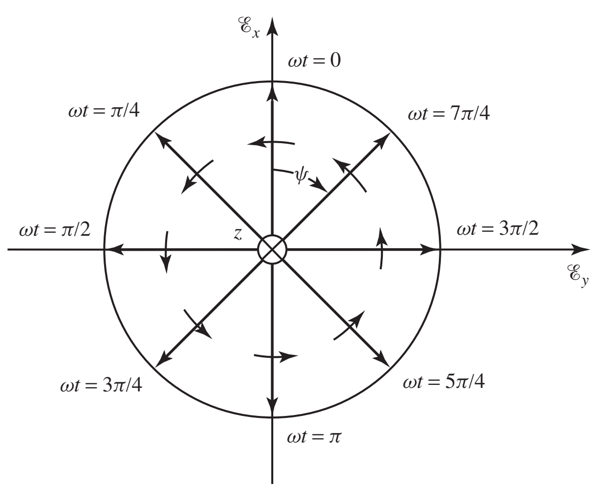


Figure 4-15 Left-hand circularly polarized wave.

The instantaneous electric field vector can be written as

$$\begin{aligned}\boldsymbol{\mathscr{E}} &= \text{Re}\left[\hat{\mathbf{a}}_x E_{\text{L}} e^{j(\omega t-\beta z)} + \hat{\mathbf{a}}_y E_{\text{L}} e^{j(\omega t-\beta z+\pi/2)}\right] \\ &= E_{\text{L}}\text{Re}\left\{\left[\hat{\mathbf{a}}_x + j\hat{\mathbf{a}}_y\right] e^{j(\omega t-\beta z)}\right\} \end{aligned} \tag{4-53d}$$

In (4-53d) we note a 90° phase advance of the $\mathscr{E}_y$ component relative to the $\mathscr{E}_x$ component.

Example 4-8

Determine the polarization and sense of rotation of the wave given by (4-50a) if $\phi_x = -\pi/2$, $\phi_y = 0$, and $E^+_{x0} = E^+_{y0} = E_{\text{L}}$.

Solution: Since

$$\phi_x = -\frac{\pi}{2}$$
$$\phi_y = 0$$
$$E^+_{x0} = E^+_{y0} = E_{\text{L}}$$

then

$$\mathscr{E}_x = E_{\text{L}} \cos\left(\omega t - \frac{\pi}{2}\right) = E_{\text{L}} \sin(\omega t)$$
$$\mathscr{E}_y = E_{\text{L}} \cos(\omega t)$$

and the locus of the amplitude is

$$\mathscr{E} = \sqrt{\mathscr{E}_x^2 + \mathscr{E}_y^2} = \sqrt{E_{\text{L}}^2(\sin^2 \omega t + \cos^2 \omega t)} = E_{\text{L}}$$

The angle ψ is given by

$$\psi = \tan^{-1}\left[\frac{\mathscr{E}_y}{\mathscr{E}_x}\right] = \tan^{-1}\left[\frac{E_{\text{L}}\cos(\omega t)}{E_{\text{L}}\sin(\omega t)}\right] = \tan^{-1}[\cot(\omega t)] = \frac{\pi}{2} - \omega t$$

The locus of the electric field vector is a circle of radius E_{L}, and it rotates counterclockwise with an angular frequency ω as shown in Figure 4-16; *hence, it is a left-hand circular polarization*. For this case we can write the electric field as

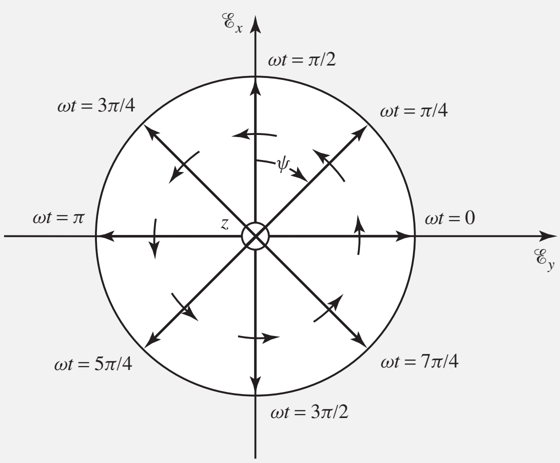


Figure 4-16 Left-hand circularly polarized wave.

$$\mathscr{E} = \mathrm{Re}\left[\hat{\mathbf{a}}_x E_{\mathrm{L}} e^{j(\omega t - \beta z - \pi/2)} + \hat{\mathbf{a}}_y E_{\mathrm{L}} e^{j(\omega t - \beta z)}\right]$$
$$= E_{\mathrm{L}} \mathrm{Re}\left\{\left[-j\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y\right] e^{j(\omega t - \beta z)}\right\}$$

and we note a 90° phase delay of the $\mathscr{E}_x$ component relative to $\mathscr{E}_y$.

From the previous discussion we see that *left-hand circular polarization* can be achieved *if and only if its two orthogonal components have equal amplitudes and odd multiples of* 90° *phase difference of one component relative to the other. The sense of rotation* (*counterclockwise here*) *is determined by rotating the phase-leading component* (*in this instance* $\mathscr{E}_y$) *toward the phase-lagging component* (*in this instance* $\mathscr{E}_x$). *The field rotation must be viewed as the wave travels away from the observer*.

The *necessary and sufficient* conditions for circular polarization are the following:

1. The field must have two orthogonal linearly polarized components.
2. The two components must have the same magnitude.
3. The two components must have a time-phase difference of odd multiples of 90°.

The sense of rotation is always determined by rotating the phase-leading component toward the phase-lagging component and observing the field rotation as the wave is traveling away from the observer. The rotation of the phase-leading component toward the phase-lagging component should be done along the angular separation between the two components that is less than 180°. *Phases equal to or greater than* 0° *and less than* 180° *should be considered leading whereas those equal to or greater than* 180° *and less than* 360° *should be considered lagging*.

4.4.3 Elliptical Polarization

A wave is said to be elliptically polarized if the tip of the electric field vector traces, as a function of time, an elliptical locus in space. At various instants of time the electric field vector changes continuously with time in such a manner as to describe an elliptical locus. It is right-hand elliptically polarized if the electric field vector of the ellipse rotates clockwise, and it is left-hand elliptically polarized if the electric field vector of the ellipse rotates counterclockwise [8, 10–14].

Let us examine the locus of the instantaneous electric field vector ($\mathscr{E}$) at the $z = 0$ plane at all times. For this particular example, let in (4-50a)

$$\begin{aligned} \phi_x &= \frac{\pi}{2} \\ \phi_y &= 0 \\ E_{x0}^{+} &= (E_{\mathrm{R}} + E_{\mathrm{L}}) \\ E_{y0}^{+} &= (E_{\mathrm{R}} - E_{\mathrm{L}}) \end{aligned} \tag{4-54}$$

Then,

$$\begin{aligned} \mathscr{E}_x &= (E_{\mathrm{R}} + E_{\mathrm{L}}) \cos\left(\omega t + \frac{\pi}{2}\right) = -(E_{\mathrm{R}} + E_{\mathrm{L}}) \sin \omega t \\ \mathscr{E}_y &= (E_{\mathrm{R}} - E_{\mathrm{L}}) \cos(\omega t) \end{aligned} \tag{4-54a}$$

We can write the locus for the amplitude of the electric field vector as

$$\begin{aligned}
\mathscr{E}^2 = \mathscr{E}_x^2 + \mathscr{E}_y^2 &= (E_\mathrm{R} + E_\mathrm{L})^2 \sin^2 \omega t + (E_\mathrm{R} - E_\mathrm{L})^2 \cos^2 \omega t \\
&= E_\mathrm{R}^2 \sin^2 \omega t + E_\mathrm{L}^2 \sin^2 \omega t + 2E_\mathrm{R}E_\mathrm{L} \sin^2 \omega t \\
&\quad + E_\mathrm{R}^2 \cos^2 \omega t + E_\mathrm{L}^2 \cos^2 \omega t - 2E_\mathrm{R}E_\mathrm{L} \cos^2 \omega t \\
\mathscr{E}_x^2 + \mathscr{E}_y^2 &= E_\mathrm{R}^2 + E_\mathrm{L}^2 + 2E_\mathrm{R}E_\mathrm{L}\left[\sin^2 \omega t - \cos^2 \omega t\right]
\end{aligned} \tag{4-54b}$$

However,

$$\begin{aligned}
\sin \omega t &= -\mathscr{E}_x/(E_\mathrm{R} + E_\mathrm{L}) \\
\cos \omega t &= \mathscr{E}_y/(E_\mathrm{R} - E_\mathrm{L})
\end{aligned} \tag{4-54c}$$

Substituting (4-54c) into (4-54b) reduces to

$$\left\{\frac{\mathscr{E}_x}{E_\mathrm{R} + E_\mathrm{L}}\right\}^2 + \left\{\frac{\mathscr{E}_y}{E_\mathrm{R} - \mathrm{E}_\mathrm{L}}\right\}^2 = 1 \tag{4-54d}$$

which is the equation for an ellipse with the major axis $|\mathscr{E}|_{\max} = |E_\mathrm{R} + E_\mathrm{L}|$ and the minor axis $|\mathscr{E}|_{\min} = |E_\mathrm{R} - E_\mathrm{L}|$. As time elapses, the electric vector rotates and its length varies with its tip tracing an ellipse, as shown in Figure 4-17. The maximum and minimum lengths of the electric vector are the major and minor axes, given by

$$|\boldsymbol{\mathscr{E}}|_{\max} = |E_\mathrm{R} + E_\mathrm{L}|, \quad \text{when } \omega t = (2n+1)\frac{\pi}{2}, n = 0, 1, 2, \ldots \tag{4-54e}$$

$$|\boldsymbol{\mathscr{E}}|_{\min} = |E_\mathrm{R} - E_\mathrm{L}|, \quad \text{when } \omega t = n\pi, n = 0, 1, 2, \ldots \tag{4-54f}$$

The axial ratio (AR) is defined to be the ratio of the major axis (including its sign) of the polarization ellipse to the minor axis, or

$$\mathrm{AR} = -\frac{\mathscr{E}_{\max}}{\mathscr{E}_{\min}} = -\frac{2(E_\mathrm{R} + E_\mathrm{L})}{2(E_\mathrm{R} - E_\mathrm{L})} = -\frac{(E_\mathrm{R} + E_\mathrm{L})}{(E_\mathrm{R} - E_\mathrm{L})} \tag{4-54g}$$

where E_R and E_L are positive real quantities. As defined in (4-54g), the axial ratio AR can take positive (for left-hand polarization) or negative (for right-hand polarization) values in the range $1 \le |\mathrm{AR}| \le \infty$. The instantaneous electric field vector can be written as

$$\begin{aligned}
\boldsymbol{\mathscr{E}} &= \mathrm{Re}\left\{\hat{\mathbf{a}}_x\left[E_\mathrm{R} + E_\mathrm{L}\right] e^{j(\omega t - \beta z + \pi/2)} + \hat{\mathbf{a}}_y\left[E_\mathrm{R} - E_\mathrm{L}\right] e^{j(\omega t - \beta z)}\right\} \\
&= \mathrm{Re}\left\{\left[\hat{\mathbf{a}}_x j(E_\mathrm{R} + E_\mathrm{L}) + \hat{\mathbf{a}}_y(E_\mathrm{R} - E_\mathrm{L})\right] e^{j(\omega t - \beta z)}\right\} \\
\boldsymbol{\mathscr{E}} &= \mathrm{Re}\left\{\left[E_\mathrm{R}(j\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y) + E_\mathrm{L}(j\hat{\mathbf{a}}_x - \hat{\mathbf{a}}_y)\right] e^{j(\omega t - \beta z)}\right\}
\end{aligned} \tag{4-54h}$$

From (4-54h) we see that we can represent an elliptical wave as the sum of a right-hand [first term of (4-54h)] and a left-hand [second term of (4-54h)] circularly polarized waves with amplitudes E_R and E_L, respectively. If $E_\mathrm{R} > \mathrm{E}_L$, the axial ratio will be negative and the right-hand circular component will be stronger than the left-hand circular component. Thus, the electric vector rotate in the same direction as that of the right-hand circularly polarized wave, producing a *right-hand elliptically polarized wave*, as shown in Figure 4-17*a*. If $E_\mathrm{L} > E_\mathrm{R}$, the axial ratio will be positive and the left-hand circularly polarized component will be stronger than the right-hand circularly polarized component. The electric field vector will rotate in the same direction as that of the left-hand circularly polarized component, producing a *left-hand elliptically polarized wave*, as shown in Figure 4-17*b*. The sign of the axial ratio carries information on the direction of rotation of the electric field vector.

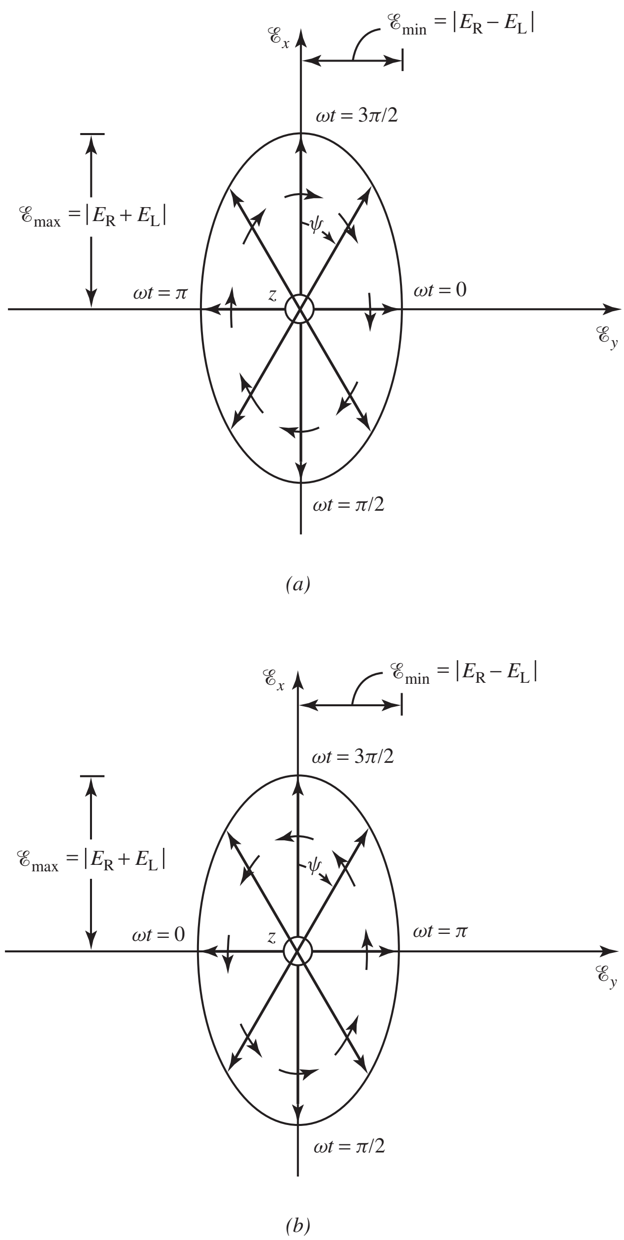


Figure 4-17 Right- and left-hand elliptical polarizations with major axis along the x axis. (a) Right-hand (clockwise) when $E_R > E_L$. (b) Left-hand (counterclockwise) when $E_R < E_L$.

An analogous situation exists when

$$\begin{aligned} \phi_x &= \frac{\pi}{2} \\ \phi_y &= 0 \\ E_{x_0}^+ &= (E_R - E_L) \\ E_{y_0}^+ &= (E_R + E_L) \end{aligned} \tag{4-55}$$

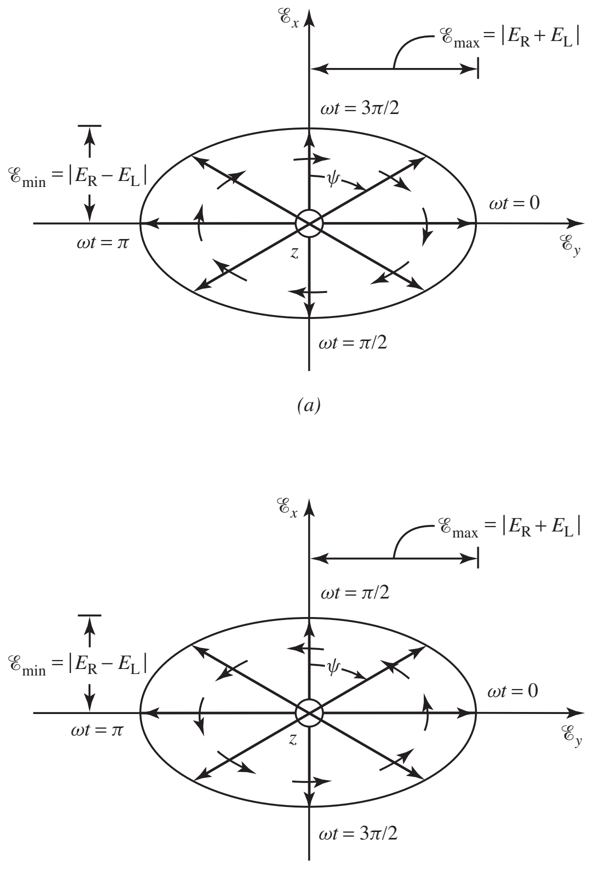


Figure 4-18 Right- and left-hand elliptical polarizations with major axis along the y axis. (*a*) Right-hand (clockwise) when $E_R > E_L$. (*b*) Left-hand (counterclockwise) when $E_R < E_L$.

The polarization loci are shown in Figure 4-18*a* and 4-18*b* when $E_R > E_L$ and $E_R < E_L$, respectively.

From (4-54e) and (4-54f), it can be seen that the component of $\boldsymbol{\mathscr{E}}$ measured along the major axis of the polarization ellipse is 90° out of phase with the component of $\boldsymbol{\mathscr{E}}$ measured along the minor axis. Also with the aid of (4-54b), it can be shown that the electric vector rotates through 90° in space between the instants of time given by (4-54e) and (4-54f) when the vector has maximum and minimum lengths, respectively. Thus the major and minor axes of the polarization ellipse are orthogonal in space, just as we might anticipate.

Since linear polarization is a special kind of elliptical polarization, we can represent a linear polarization as the sum of a right- and a left-hand circularly polarized components of *equal amplitudes*. We see that for this case ($E_R = E_L$), (4-54h) will degenerate into a linear polarization.

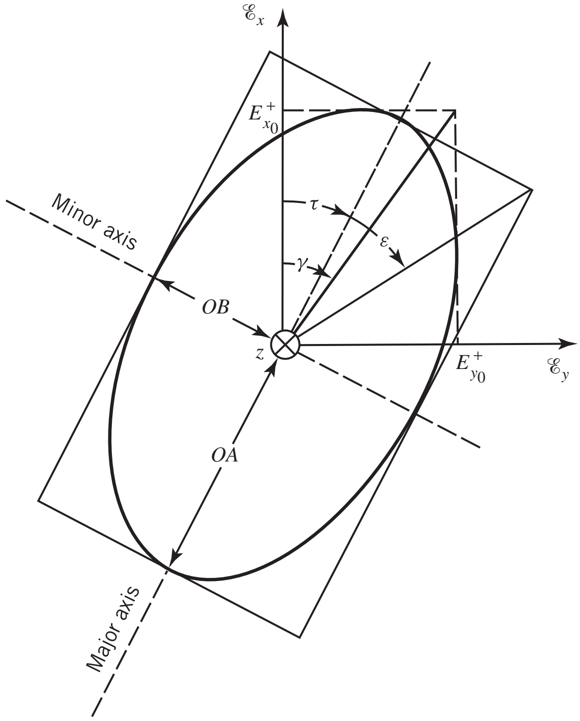


Figure 4-19 Rotation of a plane electromagnetic wave and its tilted ellipse at $z = 0$ as a function of time.

A more general orientation of an elliptically polarized locus is the tilted ellipse of Figure 4-19. This is representative of the fields of (4-50a) when

$$\Delta\phi = \phi_x - \phi_y \neq \frac{n\pi}{2} \qquad n = 0, 2, 4 \ldots$$

$$\geq 0 \quad \begin{cases} \text{for CW if } E_R > E_L \\ \text{for CCW if } E_R < E_L \end{cases} \tag{4-56a}$$

$$\leq 0 \quad \begin{cases} \text{for CW if } E_R < E_L \\ \text{for CCW if } E_R > E_L \end{cases} \tag{4-56b}$$

$$E^+_{x_0} = E_R + E_L$$

$$E^+_{y_0} = E_R - E_L \tag{4-56c}$$

Thus the major and minor axes of the ellipse do not, in general, coincide with the principal axes of the coordinate system unless the magnitudes are not equal and the phase difference between the two orthogonal components is equal to odd multiples of $\pm 90^\circ$.

The ratio of the major to the minor axes, which is defined as the axial ratio (AR), is equal to [8]

$$\mathrm{AR} = \pm\frac{\text{major axis}}{\text{minor axis}} = \pm\frac{OA}{OB}, \qquad 1 \le |\mathrm{AR}| \le \infty \tag{4-57}$$

where

$$OA = \left[\frac{1}{2}\left\{(E_{x0}^{+})^2 + (E_{y0}^{+})^2 + \left[(E_{x0}^{+})^4 + (E_{y0}^{+})^4 + 2(E_{x0}^{+})^2(E_{y0}^{+})^2\cos(2\Delta\phi)\right]^{1/2}\right\}\right]^{1/2} \tag{4-57a}$$

$$OB = \left[\frac{1}{2}\left\{(E_{x0}^{+})^2 + (E_{y0}^{+})^2 - \left[(E_{x0}^{+})^4 + (E_{y0}^{+})^4 + 2(E_{x0}^{+})^2(E_{y0}^{+})^2\cos(2\Delta\phi)\right]^{1/2}\right\}\right]^{1/2} \tag{4-57b}$$

E_{x0}^{+} and E_{y0}^{+} are given by (4-56c). The plus (+) sign in (4-57) is for left-hand and the minus (−) sign is for right-hand polarization.

The tilt of the ellipse, *relative to the x axis*, is represented by the angle τ given by

$$\tau = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left[\frac{2E_{x0}^{+}E_{y0}^{+}}{(E_{x0}^{+})^2 - (E_{y0}^{+})^2}\cos(\Delta\phi)\right] \tag{4-57c}$$

4.4.4 Poincaré Sphere

The polarization state, defined here as P, of any wave can be uniquely represented by a point on the surface of a sphere [15–19]. This is accomplished by either of the two pairs of angles (γ, δ) or (ε, τ). By referring to (4-50a) and Figure 4-20a, we can define the two pairs of angles:

$\underline{(\gamma, \delta)\ \text{set}}$

$$\gamma = \tan^{-1}\left[\frac{E_{y0}^{+}}{E_{x0}^{+}}\right] \quad \text{or} \quad \gamma = \tan^{-1}\left[\frac{E_{x0}^{+}}{E_{y0}^{+}}\right], \qquad 0^\circ \le \gamma \le 90^\circ \tag{4-58a}$$

$$\delta = \phi_y - \phi_x = \text{phase difference between } \mathscr{E}_y \text{ and } \mathscr{E}_x, \quad -180^\circ \le \delta \le 180^\circ \tag{4-58b}$$

where 2γ is the great-circle angle drawn from a reference point on the equator and δ is the equator to great-circle angle;

$\underline{(\varepsilon, \tau)\ \text{set}}$

$$\varepsilon = \cot^{-1}(\mathrm{AR}) \Rightarrow \mathrm{AR} = \cot(\varepsilon), \quad -45^\circ \le \varepsilon \le +45^\circ \tag{4-59a}$$

$$\tau = \text{tilt angle}, \qquad 0^\circ \le \tau \le 180^\circ \tag{4-59b}$$

where

$$2\varepsilon = \text{latitude}$$

$$2\tau = \text{longitude}$$

In (4-58a) *the appropriate ratio is the one that satisfies the angular limits of all the Poincaré sphere angles* (*especially those of* ε). *The axial ratio AR is positive for left-hand polarization and*

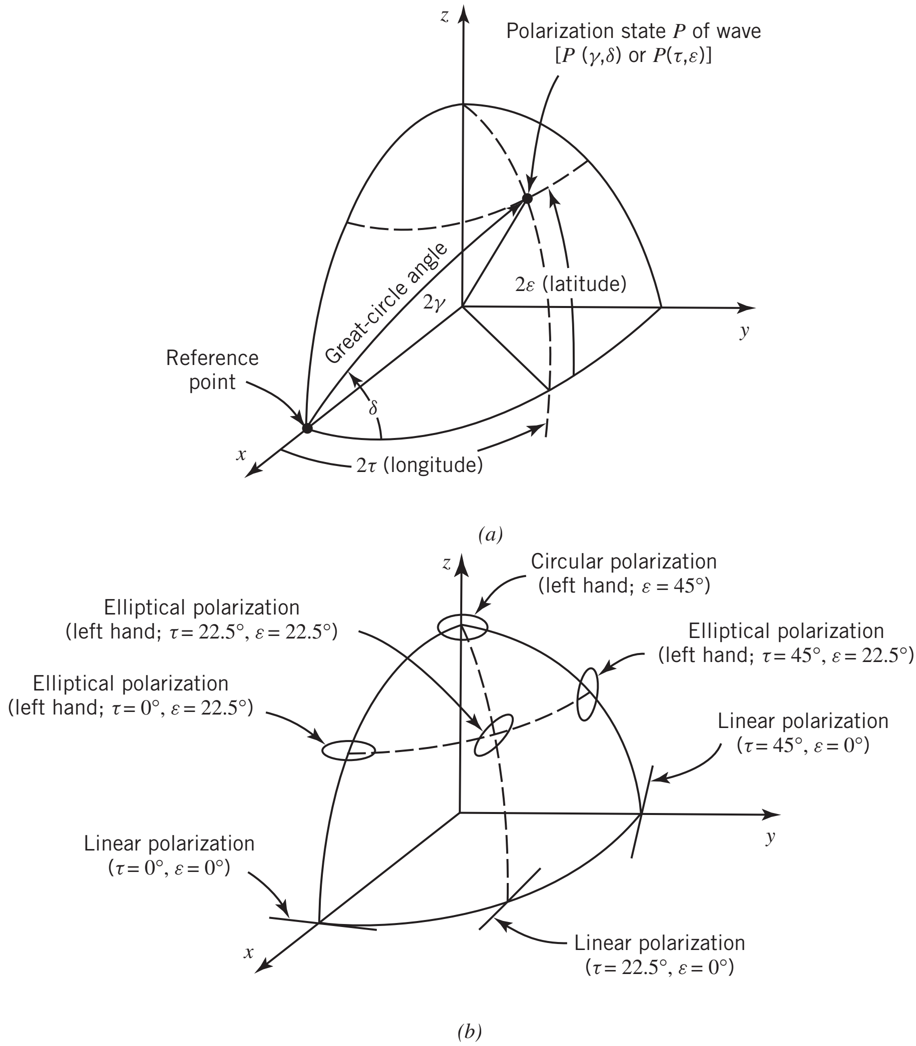


Figure 4-20 Poincaré sphere for the polarization state of an electromagnetic wave. (Source: J. D. Kraus, *Electromagnetics*, 1984, McGraw-Hill Book Co.). (*a*) Poincaré sphere. (*b*) Polarization state.

negative for right-hand polarization. Some polarization states are displayed on the first octant of the Poincaré sphere in Figure 4-20*b*. The polarization states on a planar surface representation (projection) of the Poincaré sphere ($-45° \leq \varepsilon \leq +45°$, $0° \leq \tau \leq 180°$) are shown in Figure 4-21.

For the polarization ellipse of Figure 4-19, the two sets of angles are related geometrically as shown in Figure 4-20. Analytically, it can be shown through spherical trigonometry [20] that the two pairs of angles (γ, δ) and (ε, τ) are related by

$$\boxed{\begin{aligned} \cos(2\gamma) &= \cos(2\varepsilon)\cos(2\tau) \\ \tan(\delta) &= \frac{\tan(2\varepsilon)}{\sin(2\tau)} \end{aligned}} \qquad \begin{aligned} &(4\text{-}60\text{a}) \\ &(4\text{-}60\text{b}) \end{aligned}$$

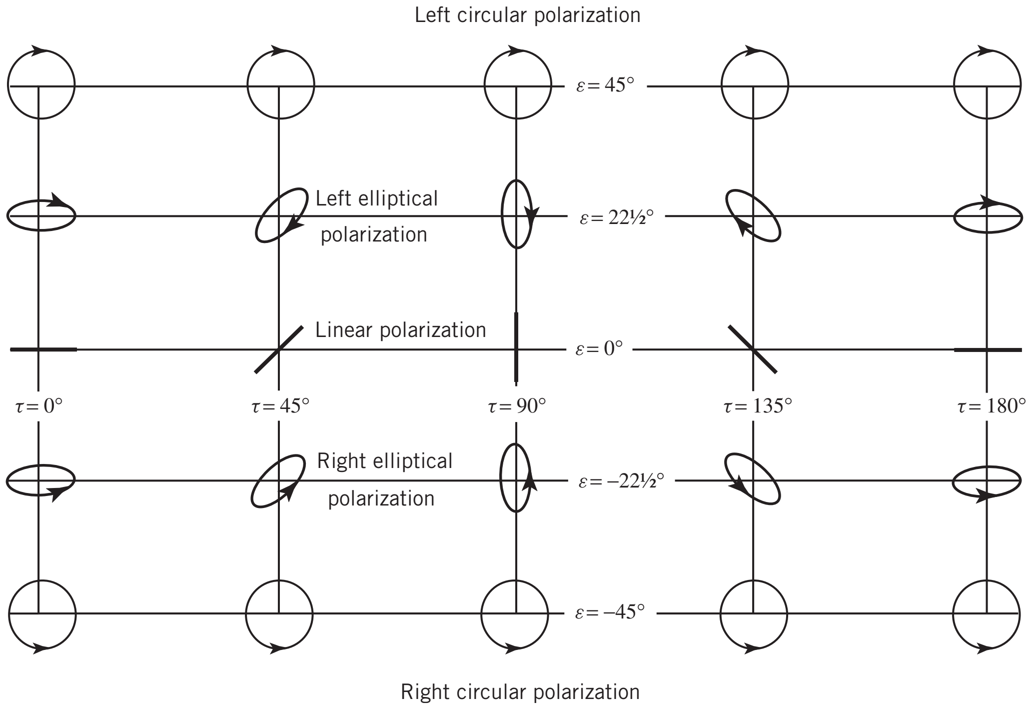


Figure 4-21 Polarization states of electromagnetic waves on a planar surface projection of a Poincaré sphere. (Source: J. D. Kraus, *Electromagnetics*, 1984, McGraw-Hill Book Co.).

or

$$\boxed{\begin{aligned} \sin(2\varepsilon) &= \sin(2\gamma)\sin(\delta) \\ \tan(2\tau) &= \tan(2\gamma)\cos(\delta) \end{aligned}} \quad \begin{aligned} &(4\text{-}61\text{a}) \\ &(4\text{-}61\text{b}) \end{aligned}$$

Thus one set can be obtained by knowing the other.

It is apparent from Figure 4-20 that the linear polarization is always found along the equator; the right-hand circular resides along the south pole and the left-hand circular along the north pole. The remaining surface of the sphere is used to represent elliptical polarization with left-hand elliptical in the upper hemisphere and right-hand elliptical on the lower hemisphere.

Because the Poincaré sphere parameter pairs (γ, δ) and (ε, τ) are related by transcendental functions, of (4-60a) and (4-60b), there may be some ambiguity at which quadrant should the angles be chosen. The angles should be selected to each satisfy respectively the range of values given by (4-58a) and (4-58b), and (4-57c), and each set should represent the same point on the Poincaré sphere. Also the range of values of the axial ratio (AR) should be $1 \leq |\text{AR}| \leq \infty$, with positive values to represent CCW (left-hand) polarization and negative values to represent CW (right-hand) polarization. A MATLAB computer program, **Polarization_Propag**, has been written and it is part of the website that accompanies this book.

Example 4-9

Determine the point on the Poincaré sphere of Figure 4-20 when the wave represented by (4-50a) is such that

$$\mathscr{E}_x = E_{x0}^+ \cos(\omega t - \beta z + \phi_x)$$
$$\mathscr{E}_y = 0$$

Solution: Using (4-58a) and (4-58b)

$$\gamma = \tan^{-1}\left[\frac{E_{y0}^+}{E_{x0}^+}\right] = \tan^{-1}\left[\frac{0}{E_{x0}^+}\right] = 0^\circ$$

and δ could be of any value, i.e., $-180^\circ \le \delta \le 180^\circ$. The values of ε and τ can now be obtained from (4-61a) and (4-61b), and they are equal to

$$2\varepsilon = \sin^{-1}\left[\sin(2\gamma)\sin(\delta)\right] = \sin^{-1}(0) = 0^\circ$$
$$2\tau = \tan^{-1}\left[\tan(2\gamma)\cos(\delta)\right] = \tan^{-1}(0) = 0^\circ$$

It is apparent that for this wave, which is obviously linearly polarized, the polarization state (point) is at the reference point of Figure 4-20. The axial ratio is obtained from (4-59a), and it is equal to

$$\text{AR} = \cot(\varepsilon) = \cot(0) = \infty$$

An axial ratio of infinity always represents linear polarization.

Example 4-10

Repeat Example 4-9 when the wave of (4-50a) is such that

$$\mathscr{E}_x = 0$$
$$\mathscr{E}_y = E_{y0}^+ \cos(\omega t - \beta z + \phi_y)$$

Solution: Using (4-58a) and (4-58b),

$$\gamma = \tan^{-1}\left[\frac{E_{y0}^+}{E_{x0}^+}\right] = \tan^{-1}(\infty) = 90^\circ$$

and δ could be of any value, i.e., $-180^\circ \le \delta \le 180^\circ$. The values of ε and τ can now be obtained from (4-61a) and (4-61b), and they are equal to

$$2\varepsilon = \sin^{-1}\left[\sin(2\gamma)\sin(\delta)\right] = \sin^{-1}(0) = 0^\circ$$
$$2\tau = \tan^{-1}\left[\tan(2\gamma)\cos(\delta)\right] = \tan^{-1}(0) = 180^\circ$$

The polarization state (point) of this linearly polarized wave is diametrically opposed to that in Example 4-9. The axial ratio is also infinity.

Example 4-11

Determine the polarization state (point) on the Poincaré sphere of Figure 4-20 when the wave of (4-50a) is such that

$$\mathscr{E}_x = E_{x0}^+ \cos(\omega t - \beta z + \phi_x) = 2E_0 \cos\left(\omega t - \beta z + \frac{\pi}{2}\right)$$

$$\mathscr{E}_y = E_{y0}^+ \cos(\omega t - \beta z + \phi_y) = E_0 \cos(\omega t - \beta z)$$

Solution: Using (4-58a) and (4-58b),

$$\gamma = \tan^{-1}\left[\frac{E_{y0}^+}{E_{x0}^+}\right] = \tan^{-1}\left[\frac{E_0}{2E_0}\right] = 26.56^\circ$$

$$\delta = \phi_y - \phi_x = -90^\circ$$

The values of ε and τ can now be obtained from (4-61a) and (4-61b), and they are equal to

$$2\varepsilon = \sin^{-1}\left[\sin(2\gamma)\sin(\delta)\right] = \sin^{-1}\left[-\sin(2\gamma)\right] = -2\gamma = -53.12^\circ$$

$$2\tau = \tan^{-1}\left[\tan(2\gamma)\cos(\delta)\right] = \tan^{-1}(0) = 0^\circ$$

Therefore, this point is situated on the principal xz plane at an angle of $2\gamma = -2\varepsilon = 53.12^\circ$ from the reference point of the x axis of Figure 4-20. The axial ratio is obtained using (4-59a), and it is equal to

$$\text{AR} = \cot(\varepsilon) = \cot(-26.56^\circ) = -2$$

The negative sign indicates that the wave has a right-hand (clockwise) polarization. Therefore the wave is right-hand elliptically polarized with AR $= -2$.

In general, points on the principal xz elevation plane, aside from the two intersecting points on the equator and the north and south poles, are used to represent elliptical polarization when the major and minor axes of the polarization ellipse of Figure 4-19 coincide with the principal axes.

If the polarization state of a wave is defined as P_w and that of an antenna as P_a, then the voltage response of the antenna due to the wave is obtained by [10, 19]

$$V = C \cos\left[\frac{P_w P_a}{2}\right] \tag{4-62}$$

where

$C =$ constant that is a function of the antenna size and field strength of the wave
$P_w =$ polarization state of the wave
$P_a =$ polarization state of the antenna
$P_w P_a =$ angle subtended by a great-circle arc from polarization P_w to P_a

Remember that the polarization of a wave, by IEEE standards [7, 8], is determined as the wave is observed from the rear (is receding). Therefore the polarization of the antenna is determined by its radiated field in the transmitting mode.

Example 4-12

If the polarization states of the wave and antenna are given, respectively, by those of Examples 4-9 and 4-10, determine the voltage response of the antenna due to that wave.

Solution: Since the polarization state P_w of the wave is at the $+x$ axis and that of the antenna P_a is at the $-x$ axis of Figure 4-20, then the angle $P_w P_a$ subtended by a great-circle arc from P_w to P_a is equal to

$$P_w P_a = 180^\circ$$

Therefore the voltage response of the antenna is, according to (4-62), equal to

$$V = C \cos\left[\frac{P_w P_a}{2}\right] = C \cos(90^\circ) = 0$$

This is expected since the fields of the wave and those of the antenna are orthogonal (cross-polarized) to each other.

Example 4-13

The polarization of a wave that impinges upon a left-hand (counterclockwise) circularly polarized antenna is circularly polarized. Determine the response of the antenna when the sense of rotation of the incident wave is

1. Left-hand (counterclockwise).
2. Right-hand (clockwise).

Solution:

1. Since the antenna is left-hand circularly polarized, its polarization state (point) on the Poincaré sphere is on the north pole ($2\gamma = \delta = 90^\circ$). When the wave is also left-hand circularly polarized, its polarization state (point) is also on the north pole ($2\gamma = \delta = 90^\circ$). Therefore, the subtended angle $P_w P_a$ between the two polarization states is equal to

   $$P_w P_a = 0^\circ$$

   and the voltage response of the antenna, according to (4-62), is equal to

   $$V = C \cos\left[\frac{P_w P_a}{2}\right] = C \cos(0) = C$$

   This represents the maximum response of the antenna, and it occurs when the polarization (including sense of rotation) of the wave is the same as that of the antenna.
2. When the sense of rotation of the wave is right-hand circularly polarized, its polarization state (point) is on the south pole ($2\gamma = 90^\circ$, $\delta = -90^\circ$). Therefore, the subtended angle $P_w P_a$ between the two polarization states is equal to

   $$P_w P_a = 180^\circ$$

   and the response of the antenna, according to (4-62), is equal to

$$V = C\cos\left[\frac{P_w P_a}{2}\right] = C\cos\left[\frac{180^\circ}{2}\right] = C\cos(90^\circ) = 0$$

This represents a null response of the antenna, and it occurs when the sense of rotation of the circularly polarized wave is opposite to that of the circularly polarized antenna. This is one technique, in addition to those shown in Example 4-12, that can be used to null the response of an antenna system.

4.5 MULTIMEDIA

On the website that accompanies this book, the following multimedia resources are included for the review, understanding and presentation of the material of this chapter.

- **MATLAB** computer programs:
  a. **Polarization_Diagram_Ellipse_Animation:** Animates the 3-D polarization diagram of a rotating electric field vector (Figure 4-8). It also animates the 2-D polarization ellipse (Figure 4-19) for linear, circular and elliptical polarized waves, and sense of rotation. It also computes the axial ratio (AR).
  b. **Polarization_Propag:** Computes the Poincaré sphere angles, and thus the polarization wave traveling in an infinite homogeneous medium.
- **Power Point (PPT)** viewgraphs, in multicolor.

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PROBLEMS

4.1. A uniform plane wave having only an x component of the electric field is traveling in the $+z$ direction in an unbounded lossless, source-free region. Using Maxwell's equations write expressions for the electric and corresponding magnetic field intensities. Compare your answers to those of (4-2b) and (4-3c).

4.2. Using Maxwell's equations, find the magnetic field components for the wave whose electric field is given in Example 4-1. Compare your answer with that obtained in the solution of Example 4-1.

4.3. The complex $\mathbf{H}$ field of a uniform plane wave, traveling in an unbounded source-free medium of free space, is given by

$$\mathbf{H} = \frac{1}{120\pi}(\hat{\mathbf{a}}_x - 2\hat{\mathbf{a}}_y)e^{-j\beta_0 z}$$

Find the:
(a) Corresponding electric field.
(b) Instantaneous power density vector.
(c) Time-average power density.

4.4. The complex $\mathbf{E}$ field of a uniform plane wave is given by

$$\mathbf{E} = (\hat{\mathbf{a}}_x + j\hat{\mathbf{a}}_z)e^{-j\beta_0 y} + (2\hat{\mathbf{a}}_x - j\hat{\mathbf{a}}_z)e^{+j\beta_0 y}$$

Assuming an unbounded source-free, free-space medium, find the:
(a) Corresponding magnetic field.
(b) Time-average power density flowing in the $+y$ direction.
(c) Time-average power density flowing in the $-y$ direction.

4.5. The magnetic field of a uniform plane wave in a source-free region is given by

$$\mathbf{H} = 10^{-6}\left[-\hat{\mathbf{a}}_x(2+j) + \hat{\mathbf{a}}_z(1+j3)\right]e^{+j\beta y}$$

Assuming that the medium is free space, determine the:
(a) Corresponding electric field.
(b) Time-average power density.

4.6. The electric field of a uniform plane wave traveling in a source-free region of free space is given by

$$\mathbf{E} = 10^{-3}(\hat{\mathbf{a}}_x + j\hat{\mathbf{a}}_y)\sin(\beta_0 z)$$

(a) Is this a traveling or a standing wave?
(b) Identify the traveling wave(s) of the electric field and the direction(s) of travel.
(c) Find the corresponding magnetic field.
(d) Determine the time-average power density of the wave.

4.7. The magnetic field of a uniform plane wave traveling in a source-free, free-space region is given by

$$\mathbf{H} = 10^{-6}(\hat{\mathbf{a}}_y + j\hat{\mathbf{a}}_z)\cos(\beta_0 x)$$

(a) Is this a traveling or a standing wave?
(b) Identify the traveling wave(s) of the magnetic field and the direction(s) of travel.
(c) Find the corresponding electric field.
(d) Determine the time-average power density of the wave.

4.8. A uniform plane wave is traveling in the $-z$ direction inside an unbounded source-free, free-space region. Assuming that the electric field has only an E_x component, its value at $z = 0$ is 4×10^{-3} V/m, and its frequency of operation is 300 MHz, write expressions for the:
(a) Complex electric and magnetic fields.
(b) Instantaneous electric and magnetic fields.
(c) Time-average and instantaneous power densities.
(d) Time-average and instantaneous electric and magnetic energy densities.

4.9. A uniform plane wave traveling inside an unbounded free-space medium has peak electric and magnetic fields given by

$$\mathbf{E} = \hat{\mathbf{a}}_x E_0 e^{-j\beta_0 z}$$
$$\mathbf{H} = \hat{\mathbf{a}}_y H_0 e^{-j\beta_0 z}$$

where $E_0 = 1$ mV/m.
(a) Evaluate H_0.
(b) Find the corresponding average power density. Evaluate all the constants.
(c) Determine the volume electric and magnetic energy densities. Evaluate all the constants.

4.10. The complex electric field of a uniform plane wave traveling in an unbounded nonferromagnetic dielectric medium is given by

$$\mathbf{E} = \hat{\mathbf{a}}_y 10^{-3} e^{-j2\pi z}$$

where z is measured in meters. Assuming that the frequency of operation is 100 MHz, find the:
(a) Phase velocity of the wave (give units).
(b) Dielectric constant of the medium.
(c) Wavelength (in meters).
(d) Time-average power density.
(e) Time-average total energy density.

4.11. The complex electric field of a time-harmonic field in free space is given by

$$\mathbf{E} = \hat{\mathbf{a}}_z 10^{-3}(1 + j)e^{-j(2/3)\pi x}$$

Assuming the distance x is measured in meters, find the:
(a) Wavelength (in meters).
(b) Frequency.
(c) Associated magnetic field.

4.12. A uniform plane wave is traveling inside the earth, which is assumed to be a perfect dielectric infinite in extent. If the relative permittivity of the earth is 9, find, at a frequency of 1 MHz, the:
(a) Phase velocity.
(b) Wave impedance.
(c) Intrinsic impedance.
(d) Wavelength of the wave inside the earth.

4.13. An 11-GHz transmitter radiates its power isotropically in a free-space medium. Assuming its total radiated power is 50 mW, at a distance of 3 km, find the:
(a) Time-average power density.
(b) RMS electric and magnetic fields.
(c) Total time-average volume energy densities.
In all cases, specify the units.

4.14. The electric field of a time-harmonic wave traveling in free space is given by

$$\mathbf{E} = \hat{\mathbf{a}}_x 10^{-4}(1 + j)e^{-j\beta_0 z}$$

Find the amount of real power crossing a rectangular aperture whose cross section is perpendicular to the z axis. The area of the aperture is $20\,\text{cm}^2$.

4.15. The following complex electric field of a time-harmonic wave traveling in a source-free, free-space region is given by

$$\mathbf{E} = 5 \times 10^{-3}(4\hat{\mathbf{a}}_y + 3\hat{\mathbf{a}}_z)e^{j(6y-8z)}$$

Assuming y and z represent their respective distances in meters, determine the:
(a) Angle of wave travel (relative to the z axis).
(b) Three phase constants of the wave along its oblique direction of travel, the y axis, and the z axis (in radians per meter).
(c) Three wavelengths of the wave along its oblique direction of travel, the y axis, and the z axis (in meters).
(d) Three phase velocities of the wave along the oblique direction of travel, the y axis, and the z axis (in meters per second).
(e) Three energy velocities of the wave along the oblique direction of travel, the y axis, and the z axis (in meters per second).
(f) Frequency of the wave.
(g) Associated magnetic field.

4.16. Using Maxwell's equations, determine the magnetic field of (4-18b) given the electric field of (4-18a).

4.17. Given the electric field of Example 4-2 and using Maxwell's equations, determine the magnetic field. Compare it with that found in the solution of Example 4-2.

4.18. Given (4-19a) and (4-19c), determine the phase velocities of (4-22) and (4-23).

4.19. Derive the energy velocity of (4-24) using the definition of (4-9), (4-18a), and (4-18b).

4.20. A uniform plane wave of 3 GHz is incident upon an unbounded conducting medium of copper that has a conductivity of 5.76×10^7 S/m, $\varepsilon = \varepsilon_0$, and $\mu = \mu_0$. Find the approximate:
(a) Intrinsic impedance of copper.
(b) Skin depth (in meters).

4.21. The magnetic field intensity of a plane wave traveling in a lossy earth is given by

$$\mathbf{H} = (\hat{\mathbf{a}}_y + j2\hat{\mathbf{a}}_z)H_0 e^{-\alpha x} e^{-j\beta x}$$

where $H_0 = 1\ \mu$A/m. Assuming the lossy earth has a conductivity of 10^{-4} S/m, a dielectric constant of 9, and the frequency of operation is 1 GHz, find inside the earth the:
(a) Corresponding electric field vector.
(b) Average power density vector.
(c) Phase constant (radians per meter).
(d) Phase velocity (meters per second).
(e) Wavelength (meters).
(f) Attenuation constant (Nepers per meter).
(g) Skin depth (meters).

4.22. Sea water is an important medium in communication between submerged submarines or between submerged submarines and receiving and transmitting stations located above the surface of the sea. Assuming the constitutive electrical parameters of the sea are $\sigma = 4$ S/m, $\varepsilon_r = 81$, $\mu_r = 1$, and $f = 10^4$ Hz, find the:
(a) Complex propagation constant (per meter).
(b) Phase velocity (meters per second).
(c) Wavelength (meters).
(d) Attenuation constant (Nepers per meter).
(e) Skin depth (meters).

4.23. The electrical constitutive parameters of moist earth at a frequency of 1 MHz are $\sigma = 10^{-1}$ S/m, $\varepsilon_r = 4$, and $\mu_r = 1$. Assuming that the electric field of a uniform plane wave at the interface (on the side of the earth) is 3×10^{-2} V/m, find the:
(a) Distance through which the wave must travel before the magnitude of the electric field reduces to 1.104×10^{-2} V/m.
(b) Attenuation the electric field undergoes in part (a) (in decibels).
(c) Wavelength inside the earth (in meters).
(d) Phase velocity inside the earth (in meters per second).
(e) Intrinsic impedance of the earth.

4.24. The complex electric field of a uniform plane wave is given by

$$\mathbf{E} = 10^{-2}\left[\hat{\mathbf{a}}_x\sqrt{2} + \hat{\mathbf{a}}_z(1+j)e^{j\pi/4}\right]e^{-j\beta y}$$

(a) Find the polarization of the wave (linear, circular, or elliptical).
(b) Determine the sense of rotation (clockwise or counterclockwise).
(c) Sketch the figure the electric field traces as a function of ωt.

4.25. The complex magnetic field of a uniform plane wave is given by

$$\mathbf{H} = \frac{10^{-3}}{120\pi}(\hat{\mathbf{a}}_x - j\hat{\mathbf{a}}_z)e^{+j\beta y}$$

(a) Find the polarization of the wave (linear, circular, or elliptical).
(b) State the direction of rotation (clockwise or counterclockwise). Justify your answer.
(c) Sketch the polarization curve denoting the $\boldsymbol{\mathscr{H}}$-field amplitude, and direction of rotation. Indicate on the curve the various times for the rotation of the vector.

4.26. In a source-free, free-space region, the complex magnetic field of a time-harmonic field is represented by

$$\mathbf{H} = \left[\hat{\mathbf{a}}_x(1+j) + \hat{\mathbf{a}}_z\sqrt{2}e^{j\pi/4}\right]\frac{E_0}{\eta_0}e^{-j\beta_0 y}$$

where E_0 is a constant and η_0 is the intrinsic impedance of free space. Determine the:
(a) Polarization of the wave (linear, circular, or elliptical). Justify your answer.
(b) Sense of rotation, if any.
(c) Corresponding electric field.

4.27. Show that any linearly polarized wave can be decomposed into two circularly polarized waves (one CW and the other CCW) but both traveling in the same direction as the linearly polarized wave.

4.28. The electric field of a $f = 10$ GHz time-harmonic uniform plane wave traveling in a perfect dielectric medium is given by

$$\mathbf{E} = \left(\hat{\mathbf{a}}_x + j2\hat{\mathbf{a}}_y\right)e^{-j600\pi z}$$

where z is in meters. Determine, assuming the permeability of the medium is the same as that of free space, the:
(a) Wavelength of the wave (in meters).
(b) Velocity of the wave (in meters/sec).
(c) Dielectric constant (relative permittivity) of the medium (dimensionless).
(d) Intrinsic impedance of the medium (in ohms).
(e) Wave impedance of the medium (in ohms).
(f) Vector magnetic field of the wave.
(g) Polarization of the wave (linear, circular, elliptical; AR; and sense of rotation).

4.29. The spatial variations of the electric field of a time-harmonic wave traveling in free space are given by

$$\mathbf{E}(x) = \hat{\mathbf{a}}_y e^{-j(\beta_0 x - \frac{\pi}{4})} + \hat{\mathbf{a}}_z e^{-j(\beta_0 x - \frac{\pi}{2})}$$

Determine, using the necessary and sufficient conditions of the wave, the:
(a) Direction of wave travel ($+x$, $-x$, $+y$, $-y$, $+z$ or $-z$) based on $e^{+j\omega t}$ time.
(b) Polarization of the wave (linear, circular or elliptical). Justify your answer.
(c) Sense of rotation (CW or CCW), if any, of the wave. Justify your answer.

4.30. The spatial variations of the electric field of a time-harmonic wave traveling in free space are given by

$$\mathbf{E}(z) = \hat{\mathbf{a}}_x 2e^{-j(\beta_0 z - \frac{\pi}{4})} + \hat{\mathbf{a}}_y e^{-j(\beta_0 z - \frac{3\pi}{4})}$$

Determine the:
(a) Direction of wave travel ($+x$, $-x$, $+y$, $-y$, $+z$ or $-z$) based on $e^{+j\omega t}$ time.
(b) Two pairs of Poincaré sphere polarization parameters (γ, δ) and (ε, τ).
(c) Based on either one of the two pairs of parameters from part (b), state the:
- Polarization of the wave (linear, circular or elliptical). Justify your answer.
- Sense of rotation (CW or CCW) of the wave. Justify your answer.
- Axial Ratio. Justify your answer.

4.31. The time-harmonic electric field traveling inside an infinite lossless dielectric medium is given by

$$\mathbf{E}^i(z) = \left(j2\hat{\mathbf{a}}_x + 5\hat{\mathbf{a}}_y\right) E_0 e^{-j\beta z}$$

where β and E_o are real constants. Assuming a $e^{+j\omega t}$ time convention, determine the:
(a) Polarization of the wave (linear, circular or elliptical). You must justify your answer. Be specific.
(b) Sense of rotation (CW or CCW). You must justify your answer. Be specific.
(c) Axial Ratio (AR) based on the expression of the electric field. You must justify your answer. Be specific.
(d) Poincaré sphere angles (in degrees):
- γ and δ
- ε and τ

Make sure that the polarization point on the Poincaré sphere based on the pair of angles (γ, δ) is the same as that based on the set of angles (ε, τ).
(e) Axial Ratio (AR) based on the Poincaré sphere angles. Compare with that in part (c).

4.32. In a source-free, free-space region the complex magnetic field is given by

$$\mathbf{H} = j(\hat{\mathbf{a}}_y - j\hat{\mathbf{a}}_z)\frac{E_0}{\eta_0}e^{+j\beta_0 x}$$

where E_0 is a constant and η_0 is the intrinsic impedance of free space. Find the:
(a) Polarization of the wave (linear, circular, or elliptical). Justify your answer.
(b) Sense of rotation, if any (CW or CCW). Justify your answer.
(c) Time-average power density.
(d) Polarization of the wave on the Poincaré sphere.

4.33. The electric field of a time-harmonic wave is given by

$$\mathbf{E} = 2 \times 10^{-3}(\hat{\mathbf{a}}_x + \hat{\mathbf{a}}_y)e^{-j2z}$$

(a) State the polarization of the wave (linear, circular, or elliptical).
(b) Find the polarization on the Poincaré sphere by identifying the angles δ, γ, τ and ε (in degrees).
(c) Locate the polarization point on the Poincaré sphere.

4.34. For a uniform plane wave represented by the electric field

$$\mathbf{E} = E_0(\hat{\mathbf{a}}_x - j2\hat{\mathbf{a}}_y)e^{-j\beta z}$$

where E_0 is constant, do the following.
(a) Determine the longitude angle 2τ, latitude angle 2ε, great-circle angle 2γ, and equator to great-circle angle δ (all

in degrees) that are used to identify and locate the polarization of the wave on the Poincaré sphere.

(b) Using the answers from part (a), state the polarization of the wave (linear, circular, or elliptical), its sense of rotation (CW or CCW), and its Axial Ratio.

(c) Find the signal loss (in decibels) when the wave is received by a right-hand circularly polarized antenna.

4.35. The electric field of (4-50a) has an Axial Ratio of infinity and a great-circle angle of $2\gamma = 109.47^\circ$.

(a) Find the relative magnitude (ratio) of $E_{y_0}^+$ to $E_{x_0}^+$. Which component is more dominant, E_x or E_y? Use the first definition of γ in (4-58a).

(b) Identify the polarization point on the Poincaré sphere (i.e., find δ, τ, and ε in degrees).

(c) State the polarization of the wave (linear, circular, or elliptical).

4.36. A uniform plane wave is traveling along the $+z$ axis and its electric field is given by

$$\mathbf{E}_w = (\hat{\mathbf{a}}_x + j\hat{\mathbf{a}}_y)e^{-j\beta z}E_0$$

This incident plane wave impinges upon an antenna whose field radiated along the z axis is given by

(a) $\mathbf{E}_{aa} = (\hat{\mathbf{a}}_x + j\hat{\mathbf{a}}_y)e^{+j\beta z}E_a$

(b) $\mathbf{E}_{ab} = (\hat{\mathbf{a}}_x - j\hat{\mathbf{a}}_y)e^{+j\beta z}E_a$

Determine the:

1. Polarization of the incident wave (linear, circular, elliptical; sense of rotation; and AR).
2. Polarization of antenna of part (a) (linear, circular, elliptical; sense of rotation; and AR).
3. Polarization of antenna of part (b) (linear, circular, elliptical; sense of rotation; and AR).
4. Normalized output voltage when the incident wave impinges upon the antenna whose electric field is that of part (a).
5. Normalized output voltage when the incident wave impinges upon the antenna whose electric field is that of part (b).

4.37. The field radiated by an antenna has electric field components represented by (4-50a) such that $E_{x_0}^+ = E_{y_0}^+$ and its Axial Ratio is infinity.

(a) Identify the polarization point on the Poincaré sphere (i.e., find γ, δ, τ, and ε in degrees).

(b) If this antenna is used to receive the wave of Problem 4.35, find the polarization loss (in decibels). To do this part, use the Poincaré sphere parameters.